Problem 1  
**An Additive Noise Channel**

Find the channel capacity of the following discrete memoryless channel:

\[
\begin{array}{c}
X \quad + \\
\downarrow \quad Z \quad \uparrow \\
X \quad \rightarrow \quad Y
\end{array}
\]

where \( \Pr[Z = 0] = \Pr[Z = a] = \frac{1}{2} \) for some fixed value \( a \in \mathbb{R} \). The input \( X \) takes values in the binary alphabet \( \mathcal{X} = \{0, 1\} \). Assume that \( Z \) is independent of \( X \).

*Hint: Observe that the channel capacity depends on the value of \( a \), i.e., you need to introduce a case distinction!*

Problem 2  
**Data Processing**

Let \( X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow \cdots \rightarrow X_n \) form a Markov chain, i.e.,

\[
P_{X_1,\ldots,X_n}(x_1, x_2, \ldots, x_n) = P_{X_1}(x_1)P_{X_2|X_1}(x_2|x_1)\cdots P_{X_n|X_{n-1}}(x_n|x_{n-1}).
\]

Reduce \( I(X_1; X_2, \ldots, X_n) \) to its simplest form.

Problem 3  
**Preprocessing the Output**

A communication channel with transition probabilities \( W(\cdot\mid\cdot) \) and channel capacity

\[
\mathcal{C} = \max_{P_X} I(X; Y)
\]

is given. A helpful statistician preprocesses the output by forming \( \tilde{Y} = g(Y) \). He claims that this will strictly improve the capacity.

a) Show that he is wrong.

b) Under what conditions does he not strictly decrease the capacity?
Problem 4

A Channel With Two Independent Looks at Y

Let $Y_1$ and $Y_2$ be conditionally independent given $X$.

a) Show that $I(X;Y_1,Y_2) = I(X;Y_1) + I(X;Y_2) - I(Y_1;Y_2)$.

b) Conclude that the capacity of the channel

\[
\begin{array}{c}
  X \\
  \downarrow \\
  W_1 \rightarrow Y_1 \\
  \downarrow \\
  W_2 \rightarrow Y_2
\end{array}
\]

is upper bounded by the sum of the capacity of the channel

\[
\begin{array}{c}
  X \\
  \rightarrow W_1 \rightarrow Y_1
\end{array}
\]

and the capacity of the channel

\[
\begin{array}{c}
  X \\
  \rightarrow W_2 \rightarrow Y_2
\end{array}
\]

Problem 5

Miscellaneous Capacities

Find the capacity and an optimizing input probability assignment for each of the discrete memoryless channels in Figure 1.
Figure 1: Miscellaneous channels.