Information Theory I

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http://www.isi.ee.ethz.ch/teaching/courses/it1.html

Problem 1 — *Encoder and Decoder as Part of the Channel*

Consider a binary symmetric channel (BSC) with crossover probability 0.1. A possible coding scheme for this channel with two codewords of length 3 is to encode message m_1 as 000 and m_2 as 111. A possible decoder is the majority decoder that, given three binary outputs, declares that message m_1 was sent if at least two of the output bits are 0. Otherwise it declares m_2. With this coding scheme we can consider the combination of encoder, channel, and decoder as forming a new BSC, with two possible inputs m_1 and m_2 and two possible outputs m_1 and m_2.

a) Calculate the crossover probability of the new channel.

b) What is the capacity of the new channel in $\frac{\text{bits}}{\text{use of original channel}}$?

c) What is the capacity of the original BSC with crossover probability 0.1?

d) Prove a general result that for any DMC, considering the encoder, channel, and decoder together as a new channel from messages to estimated messages will not increase the capacity in bits per transmission of the original channel.

Problem 2 — *Nonuniqueness of Capacity-Achieving Input Distributions*

a) Give an example of a DMC whose capacity is achieved by two different input distributions. That is, find a channel $W(y|x)$ and two different input distributions Q_1 and Q_2 such that

$$\mathsf{C} = I(Q_1, W) = I(Q_2, W).$$

b) Show that for any DMC, all capacity-achieving input distributions induce the same the output distribution. In other words, show that if two input distributions Q_1 and Q_2 satisfy

$$\mathsf{C} = I(Q_1, W) = I(Q_2, W),$$

then

$$(Q_1 W)(y) = (Q_2 W)(y) \quad \forall y \in \mathcal{Y},$$

where

$$(Q_\nu W)(y) = \sum_{x \in \mathcal{X}} Q_\nu(x) W(y|x), \qquad \nu = 1, 2.$$

Problem 3 — *The Binary Jammer Channel*

Consider the problem of communicating over a binary symmetric channel with random interference. For each input bit, there is a probability q that the channel is "free," in which case it behaves as an ordinary binary symmetric channel with crossover probability ϵ. With probability $1-q$, however, the channel is "blocked" by an interferer. When the channel is blocked, the interferer replaces your input bit with a random input bit drawn according to $p(0) = p(1) = 1/2$. Find the capacity C of the jammer channel under the following conditions:

a) Neither the encoder nor the decoder knows whether the channel is blocked at each time.

b) The decoder knows when the channel is blocked, but the encoder does not.

c) Show that if both encoder and decoder know the channel, any rate smaller than $q(1 - H_\mathrm{b}(\epsilon))$ is achievable. To this end, demonstrate a coding scheme that achieves this rate.

Problem 4 — *Typical Decoding vs. Maximum-Likelihood Decoding*

Consider a memoryless binary symmetric channel (BSC) with crossover probability α and a fixed codebook.

a) Given some $\epsilon > 0$, describe the joint typicality decoder with $P_X(0) = P_X(1) = \frac{1}{2}$. Be explicit in your description of the decoder.

b) Describe the maximum-likelihood (ML) decoder.

c) Which decoder gives rise to a lower average probability of message error?