Information Theory I

Fall Semester 2017

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Exercise 10 of November 22, 2017

http://www.isi.ee.ethz.ch/teaching/courses/it1.html

Problem 1 — *Channel Coding for a "Double"-Channel*

Consider two memoryless channels $W^{(1)}(y^{(1)}|x)$ and $W^{(2)}(y^{(2)}|x)$ defined over a common input alphabet $\mathcal{X}$ and over possibly different output alphabets $\mathcal{Y}^{(1)}$ and $\mathcal{Y}^{(2)}$. Let Q be some input distribution on $\mathcal{X}$ and let

$$\mathsf{R} < \min\bigl\{I\bigl(Q, W^{(1)}\bigr), I\bigl(Q, W^{(2)}\bigr)\bigr\},$$

so that R is smaller than the mutual information corresponding to the input distribution Q on each channel. Given any $\epsilon > 0$, prove that for sufficiently large blocklength n there exists a rate-R codebook $\mathcal{C}$ and two decoders

$$\psi^{(1)}\colon \bigl(\mathcal{Y}^{(1)}\bigr)^n \to \bigl\{1, \ldots, 2^{n\mathsf{R}}\bigr\}$$
$$\psi^{(2)}\colon \bigl(\mathcal{Y}^{(2)}\bigr)^n \to \bigl\{1, \ldots, 2^{n\mathsf{R}}\bigr\}$$

such that the maximum error probability is smaller than ϵ on both channels.
Note: Each channel has a different decoder, but you must show that there exists *one* codebook that is good for both channels.

Problem 2 — *Zero-Error Capacity*

A channel with alphabet $\{0, 1, 2, 3, 4\}$ has transition probabilities of the form

$$W(y|x) = \begin{cases} \frac{1}{2} & \text{if } y = x \pm 1 \bmod 5, \\ 0 & \text{otherwise.} \end{cases}$$

a) Compute the capacity of this channel in bits.

b) The zero-error capacity of a channel is the number of bits per channel use that can be transmitted with zero probability of error. Clearly, the zero-error capacity of this pentagonal channel is at least 1 bit (transmit 0 or 1 with probability $\frac{1}{2}$). Find a block code that shows that the zero-error capacity is greater than 1 bit.

   Hint: Consider codes of length 2.

Problem 3 *An Elementary Converse for the Binary Erasure Channel*

Consider a binary erasure channel with erasure probability $\rho \in [0,1]$ together with an encoder and a decoder. Denote the blocklength by n, the message set by $\mathcal{M} = \{1, \ldots, 2^{n\mathsf{R}}\}$, the channel input sequence by X^n, the channel output sequence by Y^n, and the output of the decoder by $\hat{M}$. Let the message M be chosen uniformly at random from $\mathcal{M}$. For $i \in \{1, \ldots, n\}$, define the random variable

$$S_i \triangleq \begin{cases} 1 & \text{if } Y_i = ?, \\ 0 & \text{otherwise.} \end{cases}$$

Define a random variable E that is one if $M \neq \hat{M}$ and zero otherwise.

a) Show that for all $s^n \in \{0,1\}^n$,

$$\Pr[E = 1 | S^n = s^n] \geq \frac{2^{n\mathsf{R}} - 2^{n(1-\kappa)}}{2^{n\mathsf{R}}},$$

where $\kappa \triangleq \frac{1}{n} \sum_{i=1}^{n} s_i$.

b) Let $\delta > 0$ be fixed. Compute

$$\lim_{n \to \infty} \Pr\left[\frac{1}{n} \sum_{i=1}^{n} S_i \geq \rho - \delta\right].$$

c) Let $\mathsf{R} > 1 - \rho$ and $\epsilon > 0$ be fixed. Deduce from Parts a) and b) that for sufficiently large n,

$$\Pr[E = 1] \geq 1 - \epsilon.$$

Thus, for sufficiently large n, the average probability of error is at least $1 - \epsilon$.

Problem 4 *Average Bit-Error Probability*

Consider a combined source-channel coding scheme comprising a DMC of capacity C, a source that emits IID Ber(1/2) bits $U_1, \ldots, U_k$, an encoder that maps the k bits to n channel input symbols $X_1, \ldots, X_n$, and a decoder that maps the resulting n channel output symbols $Y_1, \ldots, Y_n$ to an estimate of the source bits $\hat{U}_1, \ldots, \hat{U}_k$. Suppose that the rate (in bits per channel use) exceeds the capacity of the channel, i.e., $k/n > \mathsf{C}$. The task of this problem is to show that

$$\frac{1}{k} \sum_{i=1}^{k} \Pr\big[U_i \neq \hat{U}_i\big] \geq H_{\mathrm{b}}^{-1}\Big(1 - \frac{n}{k}\mathsf{C}\Big) > 0, \tag{1}$$

where $H_{\mathrm{b}}^{-1}(\cdot)$ denotes the inverse of the binary entropy function restricted to $[0, 1/2]$. In other words, if $k/n > \mathsf{C}$, then the average bit-error probability cannot be made arbitrarily small.

a) Justify the following steps:

$$\begin{aligned} n\mathsf{C} &\overset{\text{(i)}}{\geq} I(U^k; \hat{U}^k) \\ &\overset{\text{(ii)}}{=} H(U^k) - H(U^k | \hat{U}^k) \\ &\overset{\text{(iii)}}{=} k - H(U^k | \hat{U}^k) \end{aligned}$$

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\begin{aligned}
&\overset{\text{(iv)}}{=} k - \sum_{i=1}^{k} H(U_i|\hat{U}^k, U^{i-1}) \\
&\overset{\text{(v)}}{\geq} k - \sum_{i=1}^{k} H(U_i|\hat{U}_i) \\
&\overset{\text{(vi)}}{\geq} k - \sum_{i=1}^{k} H_{\text{b}}\Big(\Pr[U_i \neq \hat{U}_i]\Big) \\
&= k\left(1 - \sum_{i=1}^{k} \frac{1}{k} H_{\text{b}}\Big(\Pr[U_i \neq \hat{U}_i]\Big)\right) \\
&\overset{\text{(vii)}}{\geq} k\left(1 - H_{\text{b}}\left(\sum_{i=1}^{k} \frac{1}{k} \Pr[U_i \neq \hat{U}_i]\right)\right).
\end{aligned}
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b) Use Part a) to deduce (1).