Problem 1  

**Channel Coding for a “Double”-Channel**

Consider two memoryless channels $W^{(1)}(y^{(1)}|x)$ and $W^{(2)}(y^{(2)}|x)$ defined over a common input alphabet $\mathcal{X}$ and over possibly different output alphabets $\mathcal{Y}^{(1)}$ and $\mathcal{Y}^{(2)}$. Let $Q$ be some input distribution on $\mathcal{X}$ and let

$$R < \min\{I(Q,W^{(1)}), I(Q,W^{(2)})\},$$

so that $R$ is smaller than the mutual information corresponding to the input distribution $Q$ on each channel. Given any $\epsilon > 0$, prove that for sufficiently large blocklength $n$ there exists a rate-$R$ codebook $C$ and two decoders

$$\psi^{(1)}: (\mathcal{Y}^{(1)})^n \rightarrow \{1, \ldots, 2^{nR}\}$$
$$\psi^{(2)}: (\mathcal{Y}^{(2)})^n \rightarrow \{1, \ldots, 2^{nR}\}$$

such that the maximum error probability is smaller than $\epsilon$ on both channels.

**Note:** Each channel has a different decoder, but you must show that there exists one codebook that is good for both channels.

Problem 2  

**Zero-Error Capacity**

A channel with alphabet $\{0, 1, 2, 3, 4\}$ has transition probabilities of the form

$$W(y|x) = \begin{cases} \frac{1}{2} & \text{if } y = x \pm 1 \text{ mod } 5, \\ 0 & \text{otherwise}. \end{cases}$$

a) Compute the capacity of this channel in bits.

b) The zero-error capacity of a channel is the number of bits per channel use that can be transmitted with zero probability of error. Clearly, the zero-error capacity of this pentagonal channel is at least 1 bit (transmit 0 or 1 with probability $\frac{1}{2}$). Find a block code that shows that the zero-error capacity is greater than 1 bit.

**Hint:** Consider codes of length 2.
Problem 3

**An Elementary Converse for the Binary Erasure Channel**

Consider a binary erasure channel with erasure probability $\rho \in [0, 1]$ together with an encoder and a decoder. Denote the blocklength by $n$, the message set by $\mathcal{M} = \{1, \ldots, 2^n\}$, the channel input sequence by $X^n$, the channel output sequence by $Y^n$, and the output of the decoder by $\hat{M}$. Let the message $M$ be chosen uniformly at random from $\mathcal{M}$. For $i \in \{1, \ldots, n\}$, define the random variable

$$S_i \triangleq \begin{cases} 1 & \text{if } Y_i = \?, \\ 0 & \text{otherwise.} \end{cases}$$

Define a random variable $E$ that is one if $M \neq \hat{M}$ and zero otherwise.

a) Show that for all $s^n \in \{0, 1\}^n$,

$$\text{Pr}[E = 1 | S^n = s^n] \geq \frac{2^{nR} - 2^{n(1-\kappa)}}{2^{nR}},$$

where $\kappa \triangleq \frac{1}{n} \sum_{i=1}^{n} s_i$.

b) Let $\delta > 0$ be fixed. Compute

$$\lim_{n \to \infty} \text{Pr} \left[ \frac{1}{n} \sum_{i=1}^{n} S_i \geq \rho - \delta \right].$$

c) Let $R > 1 - \rho$ and $\epsilon > 0$ be fixed. Deduce from Parts a) and b) that for sufficiently large $n$,

$$\text{Pr}[E = 1] \geq 1 - \epsilon.$$

Thus, for sufficiently large $n$, the average probability of error is at least $1 - \epsilon$.

Problem 4

**Average Bit-Error Probability**

Consider a combined source-channel coding scheme comprising a DMC of capacity $C$, a source that emits IID $\text{Ber}(1/2)$ bits $U_1, \ldots, U_k$, an encoder that maps the $k$ bits to $n$ channel input symbols $X_1, \ldots, X_n$, and a decoder that maps the resulting $n$ channel output symbols $Y_1, \ldots, Y_n$ to an estimate of the source bits $\hat{U}_1, \ldots, \hat{U}_k$. Suppose that the rate (in bits per channel use) exceeds the capacity of the channel, i.e., $k/n > C$. The task of this problem is to show that

$$\frac{1}{k} \sum_{i=1}^{k} \text{Pr}[U_i \neq \hat{U}_i] \geq H_b^{-1} \left( 1 - \frac{n}{k} C \right) > 0,$$

where $H_b^{-1}(\cdot)$ denotes the inverse of the binary entropy function restricted to $[0, 1/2]$. In other words, if $k/n > C$, then the average bit-error probability cannot be made arbitrarily small.

a) Justify the following steps:

$$nC \overset{(i)}{=} I(U^k; \hat{U}^k) \overset{(ii)}{=} H(U^k) - H(U^k | \hat{U}^k) \overset{(iii)}{=} k - H(U^k | \hat{U}^k)$$
\[
\begin{align*}
\text{(iv)} & \quad k - \sum_{i=1}^{k} H(U_i|\hat{U}^k, U^{i-1}) \\
\text{(v)} & \quad \geq k - \sum_{i=1}^{k} H(U_i|\hat{U}_i) \\
\text{(vi)} & \quad \geq k - \sum_{i=1}^{k} H_b\left(\Pr[U_i \neq \hat{U}_i]\right) \\
& \quad = k \left(1 - \sum_{i=1}^{k} \frac{1}{k} H_b\left(\Pr[U_i \neq \hat{U}_i]\right)\right) \\
\text{(vii)} & \quad \geq k \left(1 - H_b\left(\sum_{i=1}^{k} \frac{1}{k} \Pr[U_i \neq \hat{U}_i]\right)\right).
\end{align*}
\]

b) Use Part a) to deduce (1).