Problem 1

Properties of $R(D)$

a) For a fixed $P_X$ and a fixed $P_{X|X}$, we have

$$E[d'(X, \hat{X})] = E[d(X, \hat{X}) - w(X)]$$

$$(i) = E[d(X, \hat{X})] - E[w(X)]$$

$$(ii) = E[d(X, \hat{X})] - \bar{w},$$

where (i) follows from the linearity of expectation; and (ii) follows from the definition of $\bar{w}$. Thus,

$$E[d'(X, \hat{X})] \leq D \text{ if and only if } E[d(X, \hat{X})] \leq D + \bar{w},$$

$$R'(D) = \min_{P_{\hat{X}|X}: E[d'(X, \hat{X})] \leq D} I(X; \hat{X}) = \min_{P_{\hat{X}|X}: E[d(X, \hat{X})] \leq D + \bar{w}} I(X; \hat{X}) = R(D + \bar{w}).$$

b) For every $x \in \mathcal{X}$, we choose

$$w(x) = \min_{\hat{x} \in \hat{X}} d(x, \hat{x}).$$

Then, for every $x \in \mathcal{X}$,

$$\min_{\hat{x} \in \hat{X}} d'(x, \hat{x}) = \min_{\hat{x} \in \hat{X}} [d(x, \hat{x}) - w(x)] = \min_{\hat{x} \in \hat{X}} d(x, \hat{x}) - w(x) = 0.$$

Thus, there is no essential loss of generality in assuming that for every $x \in \mathcal{X}$, there exists at least one $\hat{x} \in \hat{X}$ that reproduces the source with zero distortion: if a distortion measure $d$ does not satisfy this, we can choose $w$ as above so that the modified distortion measure $d'$ satisfies the condition, and we can use Part a) to relate $R(D)$ to $R'(D)$.

c) Here, we choose $w(0) = 1$ and $w(1) = 3$. Then, $\bar{w} = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 3 = 2$ and

$$d'(x, \hat{x}) = \begin{array}{c|cc}
  x & 0 & 1 \\
  \hat{x} & 0 & 1 \\
\end{array}$$

The smallest expected distortion for $P_X$ and $d'$ is zero: $E[d'(X, \hat{X})]$ cannot be smaller than zero, and $E[d'(X, \hat{X})]$ is zero if $\hat{X}$ is deterministically zero. If $\hat{X}$ is deterministically zero, we do not need any rate, and we obtain that $R'(D) = 0$ for all $D \geq 0$. Using Part a), we conclude that $R(D) = R'(D - \bar{w}) = R'(D - 2) = 0$ for all $D \geq 2$. (An expected distortion smaller than 2 for $P_X$ and $d'$ is not possible since $E[d(X, \hat{X})]$ cannot be smaller than two.)
Problem 2  

Erasure Distortion

a) We start with the computation of the expected distortion:

\[ E[d(X, \hat{X})] = \sum_{x \in \{0,1\}} \sum_{\hat{x} \in \{0,1,?\}} P(x)P(\hat{x}|x)d(x, \hat{x}) \]

\[ = \frac{1}{2} P_{X|X}(0|0) \cdot 0 + \frac{1}{2} P_{X|X}(1|0) \cdot \infty + \frac{1}{2} P_{X|X}(?|0) \cdot 1 \]

\[ + \frac{1}{2} P_{X|X}(0|1) \cdot \infty + \frac{1}{2} P_{X|X}(1|1) \cdot 0 + \frac{1}{2} P_{X|X}(?|1) \cdot 1 \]

\[ = \frac{1}{2} P_{X|X}(?|0) + \frac{1}{2} P_{X|X}(?|1) \]

\[ = \frac{1}{2} P_{\hat{X}|X}(?|0) \cdot 0 + \frac{1}{2} P_{\hat{X}|X}(?|1) \]

where we have to set \( P_{\hat{X}|X}(1|0) = P_{\hat{X}|X}(0|1) = 0 \) since otherwise the expected distortion is infinite. Hence, to find the rate distortion function we have to minimize \( I(X; \hat{X}) \) subject to the constraint

\[ \Pr[\hat{X} = ?] \leq D. \quad (1) \]

For \( D \geq 1 \), choosing \( \hat{X} \) to be deterministically \( ? \) satisfies this constraint: \( \Pr[\hat{X} = ?] = 1 \leq D \).

Thus, \( R(D) = 0 \) for \( D \geq 1 \).

Since \( P_{X|X}(1|0) = P_{\hat{X}|X}(0|1) = 0 \), we know that if \( \hat{X} = 0 \), then \( X = 0 \) with probability one, and if \( \hat{X} = 1 \), then \( X = 1 \) with probability one. Thus,

\[ I(X; \hat{X}) = H(X) - H(X|\hat{X}) \]

\[ = 1 - \left( \Pr[\hat{X} = 0] \underbrace{H(X|\hat{X} = 0)}_{=0} + \Pr[\hat{X} = 1] \underbrace{H(X|\hat{X} = 1)}_{=0} \right) \]

\[ + \Pr[\hat{X} = ?] \underbrace{H(X|\hat{X} = ?)}_{\leq 1 \text{ bit}} \]

\[ \geq 1 - D. \]

For \( D \in [0,1] \), the inequality is met with equality for the choice

\[ P(\hat{x}|x) = \begin{cases} 
1 - D & \text{if } \hat{x} = x, \\
D & \text{if } \hat{x} = ?, \\
0 & \text{otherwise}
\end{cases} \]

because then \( \Pr[\hat{X} = ?] = D \) and \( \Pr[X = 0 | \hat{X} = ?] = \Pr[X = 1 | \hat{X} = ?] = \frac{1}{2} \).

Hence, the rate distortion function is given as follows (in bits):

\[ R(D) = \begin{cases} 
1 - D & \text{if } 0 \leq D \leq 1, \\
0 & \text{if } D > 1.
\end{cases} \]

The rate distortion region, i.e., the set of all achievable pairs \((R, D)\), is depicted in Figure 1.

![Figure 1: Rate distortion region (light gray).](image-url)
b) We use the following scheme: for every source sequence of length $n$, choose a codeword that consists of the first $n(1-D)$ source symbols followed by $nD$ question marks. This code needs $2^{n(1-D)}$ codewords and has therefore a rate of $(1-D)$ bits. The distortion achieved by this code is as follows: the first $n(1-D)$ digits have zero distortion, the rest has distortion 1. Hence, on average we get a distortion of $D$.

(For $D \geq 1$, we will only use question marks, i.e., we only use one codeword $\hat{x} = (\ldots, ?)$. This code has zero rate and achieves a distortion of 1.)

Problem 3

**Rate Distortion Function with Infinite Distortion**

For a fixed $P_{X|X}$, the expected distortion is

$$E[d(X, \hat{X})] = PN(0)P_{X|X}(0|0)d(0, 0) + PN(0)P_{X|X}(1|0)d(0, 1)$$

$$+ PN(1)P_{X|X}(0|1)d(1, 0) + PN(1)P_{X|X}(1|1)d(1, 1)$$

$$= \frac{P}{2}P_{X|X}(1|0) \cdot \infty + \frac{P}{2}P_{X|X}(0|1).$$

We need $P_{X|X}(1|0) = 0$ since otherwise the expected distortion is infinite. Let $P_{X|X}(0|1) = p$ for some $p \in [0, 1]$. Then, we have $E[d(X, \hat{X})] \leq D$ if and only if $p \leq 2D$. For $D \geq \frac{1}{2}$, we can take $\hat{X} = 0$ (which corresponds to $p = 1$); therefore $R(D) = 0$ for $D \geq \frac{1}{2}$.

We now treat the case $D \in [0, \frac{1}{2}]$. We have

$$P_{X}(1) = PN(0)\underbrace{P_{X|X}(1|0)}_{= 0} + PN(1)\underbrace{P_{X|X}(1|1)}_{= 1/2} = \frac{1-p}{2},$$

so

$$I(X; \hat{X}) = H(\hat{X}) - H(\hat{X}|X)$$

$$= H_{b}\left(\frac{1-p}{2}\right) - PN(0)\underbrace{H(\hat{X}|X = 0)}_{= 0} - PN(1)\underbrace{H(\hat{X}|X = 1)}_{= 1/2} = H_{b}(p).$$

We need to minimize $I(X; \hat{X})$ subject to $p \in [0, 2D]$. We prove by contradiction that $p = 2D$ achieves the minimum. To that end, let $p^{*} < 2D$ be a minimizer. Denote by $W_{1}$ the transition probabilities $P_{X|X}$ associated with $p^{*}$. Denote by $W_{2}$ the transition probabilities $P_{X|X}$ associated with $p = 1$. Consider the mixture $\lambda W_{1} + \hat{\lambda} W_{2}$. Denote by $D_{1}$ the expected distortion of $W_{1}$ and by $D_{2}$ the expected distortion of $W_{2}$. Since $p^{*} < 2D$, $D_{1} < D$, and we have $D_{2} = \frac{1}{2}$. The expected distortion of the mixture is $\lambda D_{1} + \hat{\lambda} D_{2}$. Thus, there exists a $\lambda < 1$ such that the expected distortion of the mixture is less than or equal to $D$. By the convexity of the mutual information in $W$, we have

$$I(P_{X}; \lambda W_{1} + \hat{\lambda} W_{2}) \leq \lambda I(P_{X}, W_{1}) + \hat{\lambda} I(P_{X}, W_{2}) \leq I(P_{X}, W_{1}).$$

The mixture satisfies the distortion constraint and leads to a smaller mutual information, so $p^{*}$ cannot be a minimizer. Thus, (2) is minimized by $p = 2D$. Combining these results, we obtain

$$R(D) = \begin{cases} H_{b}\left(\frac{1-2D}{2}\right) - \frac{1}{2}H_{b}(2D) & \text{if } 0 \leq D \leq \frac{1}{2}, \\ 0 & \text{if } D > \frac{1}{2}. \end{cases}$$

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Problem 4

Rate Distortion for Uniform Source with Hamming Distortion

First, we note that the expected distortion $E[d(X, \hat{X})]$ is equal to $\Pr[X \neq \hat{X}]$:

$$E[d(X, \hat{X})] = \sum_x p(x) \sum_{\hat{x}} p(\hat{x}|x)d(x, \hat{x})$$

$$= \sum_x p(x) \sum_{\hat{x} \neq x} p(\hat{x}|x)$$

$$= \Pr[X \neq \hat{X}].$$

Hence, to find $R(D)$ we have to minimize $I(X; \hat{X})$ subject to the constraint that $\Pr[X \neq \hat{X}] \leq D$.

First we note that $R(D) = 0$ for $D \geq (m-1)/m$, because if $\hat{X}$ is independent of $X$, then, irrespective of $P_\hat{X}$, the expected distortion equals $(m-1)/m$ and the mutual information between $X$ and $\hat{X}$ equals 0. In the following we thus consider the case where $D < (m-1)/m$. Let $P_e = \Pr[X \neq \hat{X}]$ and observe that

$$I(X; \hat{X}) = H(X) - H(X|\hat{X})$$

$$\geq (i) \quad H(X) - H_b(P_e) - P_e \log (m-1)$$

$$\geq (ii) \quad H(X) - H_b(D) - D \log (m-1)$$

$$= (iii) \quad \log m - H_b(D) - D \log (m-1),$$

where (i) follows from Fano’s inequality; (ii) holds because $P_e \leq D$ and because the function $P_e \mapsto H_b(P_e) + P_e \log (m-1)$ is nondecreasing in $P_e$ for $P_e \in [0, \frac{m-1}{m}]$ (we will show this later); and (iii) holds because $X$ is uniformly distributed. Choosing $P_{\hat{X}|X}$ to be

$$P_{\hat{X}|X}(\hat{x}|x) = \begin{cases} 
1 - D & \text{if } \hat{x} = x, \\
\frac{D}{m-1} & \text{if } \hat{x} \neq x
\end{cases}$$

satisfies $\Pr[X \neq \hat{X}] \leq D$ and achieves equality because then

$$P_X(\hat{x}) = \sum_x P_{X,\hat{X}}(x, \hat{x}) = \sum_x P_X(x)P_{\hat{X}|X}(\hat{x}|x) = \frac{1}{m}(1-D) + (m-1)\frac{D}{m(m-1)} = \frac{1}{m}$$

and

$$P_{X|\hat{X}}(x|\hat{x}) = \frac{P_{X,\hat{X}}(x, \hat{x})}{P_X(\hat{x})} = \frac{P_{\hat{X}|X}(\hat{x}|x)P_X(x)}{P_X(\hat{x})} = \frac{P_{\hat{X}|X}(\hat{x}|x)m^{-1}}{m^{-1}} = P_{\hat{X}|X}(\hat{x}|x).$$

Consequently,

$$H(X|\hat{X}) = \sum_{\hat{x}} P_X(\hat{x})H(X|\hat{X} = \hat{x})$$

$$= \sum_{\hat{x}} P_X(\hat{x}) \left[ (1-D) \log \frac{1}{1-D} + (m-1) \frac{D}{m-1} \log \frac{m-1}{D} \right]$$

$$= (1-D) \log \frac{1}{1-D} + (m-1) \frac{D}{m-1} \log \frac{m-1}{D}$$

$$= (1-D) \log \frac{1}{1-D} + D \log \frac{1}{D} + D \log (m-1)$$

$$= H_b(D) + D \log (m-1).$$

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Hence, the rate distortion function is
\[
R(D) = \begin{cases} 
\log m - H_b(D) - D \log(m - 1) & 0 \leq D \leq \frac{m-1}{m}, \\
0 & D > \frac{m-1}{m}.
\end{cases}
\]

It remains to show that the function \( P_e \mapsto H_b(P_e) + P_e \log (m - 1) \) is nondecreasing in \( P_e \) for \( P_e \in [0, \frac{m-1}{m}] \). This can be done for example by computing its derivative. An alternative way is the following concavity argument. Observe that
\[
H_b(P_e) + P_e \log (m - 1) = P_e \log \frac{1}{P_e} + (1 - P_e) \log \frac{1}{1 - P_e} + P_e \log (m - 1)
\]
\[
= P_e \log \frac{m-1}{P_e} + (1 - P_e) \log \frac{1}{1 - P_e}
\]
\[
= \log m + P_e \log \frac{(m-1)/m}{P_e} + (1 - P_e) \log \frac{1/m}{1 - P_e}
\]
\[
= \log m - D\left( (P_e, 1 - P_e) \| (\frac{m-1}{m}, \frac{1}{m}) \right).
\]

Because relative entropy is nonnegative and equal to zero if and only if both arguments are equal, \( P_e \mapsto H_b(P_e) + P_e \log (m - 1) \) is maximized for \( P_e = \frac{m-1}{m} \). Since \( P_e \mapsto H_b(P_e) + P_e \log (m - 1) \) is concave in \( P_e \) (entropy and linear functions are concave; and the sum of concave functions is concave), it is nondecreasing in \( P_e \) for \( P_e \in [0, \frac{m-1}{m}] \).