Information Theory I

Fall Semester 2017

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Exercise 12 of December 6, 2017

http://www.isi.ee.ethz.ch/teaching/courses/it1.html

Problem 1 ***Rate Distortion with Two Distortion Functions***

a) Consider a source sequence X^n drawn IID according to some distribution P_X. For $x \in \mathcal{X}$ and $\hat{x} \in \hat{\mathcal{X}}$, consider two distortion functions $d_1(x, \hat{x})$ and $d_2(x, \hat{x})$. We call a triple $(\mathsf{R}, \mathsf{D}_1, \mathsf{D}_2)$ achievable if for every $\delta > 0$ there exists some blocklength n_0 such that for every $n \geq n_0$ there exist an encoder $f^{(n)} \colon \mathcal{X}^n \to \{1, \ldots, 2^{n\mathsf{R}}\}$ and a reconstructor $g^{(n)} \colon \{1, \ldots, 2^{n\mathsf{R}}\} \to \hat{\mathcal{X}}^n$ satisfying $\mathsf{E}[d_1(X^n, \hat{X}^n)] \leq \mathsf{D}_1 + \delta$ and $\mathsf{E}[d_2(X^n, \hat{X}^n)] \leq \mathsf{D}_2 + \delta$. Define

$$\mathsf{R}^{\mathrm{I}}(\mathsf{D}_1, \mathsf{D}_2) \triangleq \min_{\substack{P_{\hat{X}|X}: \\ \mathsf{E}[d_1(X,\hat{X})] \leq \mathsf{D}_1, \\ \mathsf{E}[d_2(X,\hat{X})] \leq \mathsf{D}_2}} I(X; \hat{X}).$$

Show that for fixed D_1 and D_2, rates $\mathsf{R} > \mathsf{R}^{\mathrm{I}}(\mathsf{D}_1, \mathsf{D}_2)$ are achievable and rates $\mathsf{R} < \mathsf{R}^{\mathrm{I}}(\mathsf{D}_1, \mathsf{D}_2)$ are not achievable.

b) Now consider the case with one encoder

$$f^{(n)} \colon \mathcal{X}^n \to \{1, \ldots, 2^{n\mathsf{R}}\}$$

and two reconstructors

$$g_1^{(n)} \colon \{1, \ldots, 2^{n\mathsf{R}}\} \to \hat{\mathcal{X}}_1^n,$$
$$g_2^{(n)} \colon \{1, \ldots, 2^{n\mathsf{R}}\} \to \hat{\mathcal{X}}_2^n$$

with corresponding distortion functions $d_1(x, \hat{x}_1)$ and $d_2(x, \hat{x}_2)$, $x \in \mathcal{X}$, $\hat{x}_1 \in \hat{\mathcal{X}}_1$, $\hat{x}_2 \in \hat{\mathcal{X}}_2$. In this case, we call a triple $(\mathsf{R}, \mathsf{D}_1, \mathsf{D}_2)$ achievable if for every $\delta > 0$ there exists some blocklength n_0 such that for all $n \geq n_0$ there exist $f^{(n)}$, $g_1^{(n)}$, and $g_2^{(n)}$ satisfying $\mathsf{E}[d_1(X^n, \hat{X}_1^n)] \leq \mathsf{D}_1 + \delta$ and $\mathsf{E}[d_2(X^n, \hat{X}_2^n)] \leq \mathsf{D}_2 + \delta$. Define

$$\mathsf{R}^{\mathrm{J}}(\mathsf{D}_1, \mathsf{D}_2) \triangleq \min_{\substack{P_{\hat{X}_1,\hat{X}_2|X}: \\ \mathsf{E}[d_1(X,\hat{X}_1)] \leq \mathsf{D}_1, \\ \mathsf{E}[d_2(X,\hat{X}_2)] \leq \mathsf{D}_2}} I(X; \hat{X}_1, \hat{X}_2).$$

Show that for fixed D_1 and D_2, rates $\mathsf{R} > \mathsf{R}^{\mathrm{J}}(\mathsf{D}_1, \mathsf{D}_2)$ are achievable and rates $\mathsf{R} < \mathsf{R}^{\mathrm{J}}(\mathsf{D}_1, \mathsf{D}_2)$ are not achievable.

Hint: Use Part a).

c) Suppose $\hat{\mathcal{X}}_1 = \hat{\mathcal{X}}_2$. Show that for fixed P_X, d_1, d_2, D_1, and D_2,

$$\mathsf{R}^{\mathrm{J}}(\mathsf{D}_1, \mathsf{D}_2) \leq \mathsf{R}^{\mathrm{I}}(\mathsf{D}_1, \mathsf{D}_2).$$

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Problem 2 ***Source-Channel Separation with Feedback***

Consider an IID source $\{U_k\}$ taking value in the finite set $\mathcal{U}$, a bounded distortion function $d(u,\hat{u})$, $u \in \mathcal{U}$, $\hat{u} \in \hat{\mathcal{U}}$, and a discrete memoryless channel $W(y|x)$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$. A combined source-channel coding scheme in the presence of feedback is described as follows: the encoder is a sequence of mappings $f_i\colon (U^k, Y^{i-1}) \mapsto X_i$, and the decoder is a mapping $g\colon Y^n \mapsto \hat{U}^k$. Show that this combined source-channel coding scheme with feedback can achieve expected distortion D only if

$$\frac{n}{k}\mathsf{C} \geq \mathsf{R}(\mathsf{D}),$$

where $\mathsf{R}(\mathsf{D})$ is the rate-distortion function of the source:

$$\mathsf{R}(\mathsf{D}) = \min_{P_{\hat{U}|U}\colon\, \mathsf{E}[d(U,\hat{U})] \leq \mathsf{D}} I(U;\hat{U})$$

and C is the capacity of the channel:

$$\mathsf{C} = \max_{P_X} I(P_X, W).$$

Problem 3 ***Average Bit-Error Probability***

Consider a combined source-channel coding scheme comprising a DMC of capacity C, a source that emits IID Bernoulli(1/2) bits $U_1,\ldots,U_k$, an encoder that maps the k bits to n channel input symbols $X_1,\ldots,X_n$, and a decoder that maps the resulting n channel output symbols $Y_1,\ldots,Y_n$ to an estimate of the source bits $\hat{U}_1,\ldots,\hat{U}_k$. Suppose that the rate (in bits per channel use) exceeds the capacity of the channel, i.e., $k/n > \mathsf{C}$. Show that

$$\frac{1}{k}\sum_{i=1}^{k} \Pr\bigl[U_i \neq \hat{U}_i\bigr] \geq H_{\mathrm{b}}^{-1}\Bigl(1 - \frac{n}{k}\mathsf{C}\Bigr) > 0,$$

where $H_{\mathrm{b}}^{-1}(\cdot)$ denotes the inverse of the binary entropy function restricted to $[0,1/2]$. In other words, if $k/n > \mathsf{C}$, then the average bit-error probability cannot be made arbitrarily small.

Hint: Use Problem 2 with the distortion function $d(u,\hat{u}) = I\{u \neq \hat{u}\}$ and use the fact that for a Bernoulli(1/2) source with Hamming distortion, $\mathsf{R}(\mathsf{D}) = 1 - H_{\mathrm{b}}(\mathsf{D})$ for $\mathsf{D} \in [0,1/2]$.