Problem 1 \hspace{1cm} \textit{Large Deviations}

Let $X_1, X_2, \ldots, X_n$ be IID random variables drawn according to the geometric distribution

$$\Pr[X = i] = p^{i-1}(1 - p), \quad i = 1, 2, \ldots$$

For a given $\alpha > \frac{1}{1 - p}$, find good estimates (to first order in the exponent) of

a) $\Pr\left[\frac{1}{n}\sum_{k=1}^{n} X_k \geq \alpha\right]$; and of

b) $\Pr\left[X_1 = i \mid \frac{1}{n}\sum_{k=1}^{n} X_k \geq \alpha\right]$.

c) Evaluate Parts a) and b) for $p = \frac{1}{2}$ and $\alpha = 4$ in the limit when $n \to \infty$.

Problem 2 \hspace{1cm} \textit{Counting States}

Suppose that an atom is equally likely to be in each of six states, $X \in \{s_1, s_2, \ldots, s_6\}$. One observes $n$ atoms $X_1, X_2, \ldots, X_n$ independently drawn according to this uniform distribution. It is observed that the frequency of occurrence of state $s_1$ is twice the frequency of occurrence of state $s_2$.

a) To first order in the exponent, what is the probability of observing this event?

b) Assuming $n$ large, find the conditional distribution of the state of the first atom $X_1$, given this observation.

Problem 3 \hspace{1cm} \textit{Running Difference}

Consider some finite alphabets $\mathcal{X} \subset \mathbb{R}$ and $\mathcal{Y} \subset \mathbb{R}$. Let $X_1, X_2, \ldots, X_n$ be IID $\sim Q_1 \in \mathcal{P}(\mathcal{X})$, and $Y_1, Y_2, \ldots, Y_n$ be IID $\sim Q_2 \in \mathcal{P}(\mathcal{Y})$, where $X_1^n$ and $Y_1^n$ are independent. Find an expression for

$$\Pr\left[\frac{1}{n}\sum_{k=1}^{n} X_k - \frac{1}{n}\sum_{k=1}^{n} Y_k \geq t\right], \quad t \in \mathbb{R},$$

good to first order in the exponent. You may leave the answer in parametric form.
Problem 4

Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be IID $\sim Q \in \mathcal{P}(\mathcal{X})$.

a) Find constraints on the type $P_\mathbf{X}$ such that the sample variance satisfies

$$\frac{\mathbf{X}_n^2}{n} - (\mathbf{X}_n)^2 \geq \alpha$$

for some given $\alpha > 0$, where $\mathbf{X}_n^2 \triangleq \frac{1}{n} \sum_{k=1}^{n} X_k^2$ and $\mathbf{X}_n \triangleq \frac{1}{n} \sum_{k=1}^{n} X_k$.

b) Find an expression for $\Pr\left[\frac{\mathbf{X}_n^2}{n} - (\mathbf{X}_n)^2 \geq \alpha\right]$ good to first order in the exponent. You can leave the answer in parametric form.