Problem 1

Shannon Lower Bound for the Rate Distortion Function

Consider an \( m \)-ary source \( X \) with a distortion measure \( d(x, \hat{x}) \) that satisfies the following property: all columns of the distortion matrix are permutations of the set \( \{d_1, d_2, \ldots, d_m\} \), i.e., for a fixed \( \hat{x} \), when running through all possible values of \( x \), the distortion measure takes on each value of \( \{d_1, \ldots, d_m\} \) once. Define the function

\[
\phi(D) \triangleq \max_{p: \sum_{i=1}^{m} p_i d_i \leq D} H(p).
\]

The Shannon lower bound on the rate distortion function is proven by the following steps:

a) Show that \( \phi(D) \) is a concave function of \( D \).

b) Justify the following series of (in)equalities for \( I(X; \hat{X}) \) if \( E[d(X, \hat{X})] \leq D \):

\[
I(X; \hat{X}) = H(X) - H(X|\hat{X}) \tag{1}
\]

\[
= H(X) - \sum_{\hat{x}} p(\hat{x}) H(X|\hat{X} = \hat{x}) \tag{2}
\]

\[
\geq H(X) - \sum_{\hat{x}} p(\hat{x}) \phi(D_{\hat{x}}) \tag{3}
\]

\[
\geq H(X) - \phi \left( \sum_{\hat{x}} p(\hat{x}) D_{\hat{x}} \right) \tag{4}
\]

\[
\geq H(X) - \phi(D), \tag{5}
\]

where

\[
D_{\hat{x}} \triangleq \sum_{x} p(x|\hat{x}) d(x, \hat{x}).
\]

c) Argue that

\[
R(D) \geq H(X) - \phi(D),
\]

which is the Shannon lower bound on the rate distortion function.

d) Show that if, in addition, we assume that the source has a uniform distribution and that the rows of the distortion matrix are permutations of each other and are of length \( m \), then \( R(D) = H(X) - \phi(D) \), i.e., the lower bound is tight.
Problem 2  
Rate Distortion of a Gaussian Source  
with Special Distortion Measure

Let $d(x, \hat{x})$ be a distortion function and consider a source $X \sim Q(\cdot)$. Let $R(D)$ be the associated rate distortion function.

a) Find $\tilde{R}(D)$ in terms of $R(D)$, where $\tilde{R}(D)$ is the rate distortion function associated with the distortion $\tilde{d}(x, \hat{x}) = d(x, \hat{x}) + a$ for some constant $a \geq 0$.

b) Now define a new distortion function $d'(x, \hat{x}) = b \cdot d(x, \hat{x})$ for some constant $b > 0$. Find the associated rate distortion function $R'(D)$ in terms of $R(D)$.

c) Let now $X \sim \mathcal{N}(0, \sigma^2)$ and $d(x, \hat{x}) = 5(x - \hat{x})^2 + 3$. What is $R(D)$?

Problem 3  
Weird Distortion Measure

Consider a uniform $m$-ary source with alphabet $\mathcal{X} = \{1, 2, \ldots, m\}$ where $m$ is an even positive integer, a reproduction alphabet $\hat{\mathcal{X}} = \mathcal{X}$, and the following distortion measure:

$$d(x, \hat{x}) \triangleq 1 + \cos(\pi x) + 1 - \text{sinc}(x - \hat{x}) \quad \forall x \in \mathcal{X}, \hat{x} \in \hat{\mathcal{X}},$$  

(6)

where

$$\text{sinc}(\xi) \triangleq \begin{cases} 
\frac{\sin(\pi \xi)}{\pi \xi} & \xi \neq 0, \\
1 & \xi = 0.
\end{cases}$$

a) For the given uniform source, derive the rate distortion function $\tilde{R}(D)$ for the Hamming distortion measure

$$\tilde{d}(x, \hat{x}) \triangleq \begin{cases} 
0 & \text{if } x = \hat{x}, \\
1 & \text{otherwise}.
\end{cases}$$

Make sure that you specify the function for all values of $D \geq 0$!

b) Relate the Hamming distortion measure $\tilde{d}(x, \hat{x})$ with the measure $d(x, \hat{x})$ in (6) and use this relation to derive the rate distortion function $R(D)$ from $\tilde{R}(D)$ found in a).

Problem 4  
Adding a New Reproduction Symbol to the Distortion Measure

Let $R(D)$ be the rate distortion function for an IID process with probability mass function $Q(\cdot)$ and distortion function $d(x, \hat{x})$, $x \in \mathcal{X}$, $\hat{x} \in \hat{\mathcal{X}}$. Now suppose that we add a new reproduction symbol $\hat{x}_0$ to $\hat{\mathcal{X}}$ with associated distortion $d(x, \hat{x}_0)$, $x \in \mathcal{X}$. Does this increase or decrease $R(D)$? Please explain carefully why! (A simple “yes” or “no” will not count as a correct answer.)