Sum Rate Bound for the Binary Erasure MAC

The binary erasure MAC is given by $Y = X_1 + X_2$, where $X_1, X_2 \in \{0,1\}$ and the addition is the standard addition in $\mathbb{Z}$. In this handout, we prove that

$$\max_{P_{X_1}\cdot P_{X_2}} I(X_1, X_2; Y) = 1.5 \text{ bits}, \tag{1}$$

which implies that rate pairs $(\mathsf{R}_1, \mathsf{R}_2)$ with $\mathsf{R}_1 + \mathsf{R}_2 > 1.5$ bits cannot be achieved on the binary erasure MAC.
We introduce $p = \Pr[X_1 = 0]$ and $q = \Pr[X_2 = 0]$ and note that

$$P_Y(y) = \begin{cases} pq & \text{if } y = 0, \\ p(1-q) + (1-p)q & \text{if } y = 1, \\ (1-p)(1-q) & \text{if } y = 2. \end{cases} \tag{2}$$

We have

$$\begin{aligned} I(X_1, X_2; Y) &= H(Y) - H(Y|X_1, X_2) \\ &= H(Y) \\ &= -(pq)\log_2(pq) - \bigl(p(1-q) + (1-p)q\bigr)\log_2\bigl(p(1-q) + (1-p)q\bigr) \\ &\quad - \bigl((1-p)(1-q)\bigr)\log_2\bigl((1-p)(1-q)\bigr), \end{aligned} \tag{3}$$

where $H(Y|X_1, X_2)$ is zero because Y is a (deterministic) function of X_1 and X_2. For $p = q = \frac{1}{2}$, $I(X_1, X_2; Y) = 1.5$ bits, so we only need to show that the maximum in (1) does not exceed 1.5 bits. In other words, it is enough to show that

$$H(Y) \leq 1.5 \text{ bits} \tag{4}$$

holds for any $p \in [0,1]$ and for any $q \in [0,1]$.
We first show that (4) is satisfied in some special cases. If $p = 0$, then $H(Y) = H_\text{b}(q)$, so clearly $H(Y) \leq 1$ bit. Similarly, if $p = 1$, then $H(Y) = H_\text{b}(q)$, so $H(Y) \leq 1$ bit, too. By symmetry, the same derivations also hold for $q = 0$ and $q = 1$.
Now let $(p^*, q^*) \in [0,1]^2$ be a pair that maximizes $H(Y)$. From the preceding discussion we may assume that $p^* \notin \{0,1\}$ and $q^* \notin \{0,1\}$, so the partial derivatives $\frac{\partial H(Y)}{\partial p}$ and $\frac{\partial H(Y)}{\partial q}$, evaluated at $p = p^*$ and $q = q^*$, are well-defined and must be zero. Any linear combination of the partial derivatives also has to be zero, so

$$\begin{aligned} 0 &\overset{!}{=} (1-2p^*)\cdot \frac{\partial H(Y)}{\partial p}\bigg|_{p=p^*,\, q=q^*} - (1-2q^*)\cdot \frac{\partial H(Y)}{\partial q}\bigg|_{p=p^*,\, q=q^*} \\ &= (p^* - q^*)\log_2 \frac{p^* q^*}{(1-p^*)(1-q^*)}, \end{aligned}$$

which is satisfied if and only if $q^* = p^*$ or $q^* = 1 - p^*$.
If $q = 1 - p$, then (3) becomes

$$\begin{aligned} H(Y) &= p(1-p)\log_2 \frac{4 \cdot \frac{1}{4}}{p(1-p)} + (2p^2 - 2p + 1)\log_2 \frac{2 \cdot \frac{1}{2}}{2p^2 - 2p + 1} + p(1-p)\log_2 \frac{4 \cdot \frac{1}{4}}{p(1-p)} \\ &= \frac{3}{2} - 2\left(p - \frac{1}{2}\right)^2 - D\Big(\big(p(1-p), 2p^2 - 2p + 1, p(1-p)\big) \,\big\|\, \big(\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4}\big)\Big) \\ &\le \frac{3}{2}, \end{aligned} \tag{5}$$

so (4) is satisfied.
If $q = p$, then (2) becomes

$$P_Y(y) = \begin{cases} p^2 & \text{if } y = 0, \\ 2p(1-p) & \text{if } y = 1, \\ (1-p)^2 & \text{if } y = 2, \end{cases}$$

which implies that $H(Y)$ is symmetric in p, i.e., $H(Y)|_p = H(Y)|_{1-p}$ holds. It is then sufficient to consider only $p \ge \frac{1}{2}$. Computing the derivative of $H(Y)$ with respect to p leads to

$$\begin{aligned} \frac{\mathrm{d}H(Y)}{\mathrm{d}p} &= \frac{2}{\ln 2} \cdot \ln \frac{1-p}{p} + 4p - 2 \\ &\le \frac{2}{\ln 2}\left(\frac{1-p}{p} - 1\right) + 4p - 2 \\ &= \frac{-2(2p-1)(1 - p\ln 2)}{p \ln 2}, \end{aligned} \tag{6}$$

where the inequality holds because $\ln z \le (z - 1)$ is true for all $z > 0$. Since $1 - p\ln 2 > 0$ holds for all $p \in [0, 1]$, the RHS of (6) is smaller than zero for $p > \frac{1}{2}$, and consequently $H(Y)$ does not attain a maximum for $p > \frac{1}{2}$. By the symmetry of $H(Y)$ in p, the same holds for $p < \frac{1}{2}$. The remaining possibility, $p = \frac{1}{2}$, leads to $H(Y) = 1.5$ bits. Therefore, (4) is also satisfied in the case $q = p$.
We conclude the proof with the remark that $p = q = \frac{1}{2}$ is in fact the unique maximizer of $H(Y)$. Indeed: In the special cases $p \in \{0, 1\}$ or $q \in \{0, 1\}$, $H(Y) \le 1$ bit holds; in the case $q = 1 - p$, (5) only holds with equality if $p = \frac{1}{2}$; and in the case $q = p$, a value $p \ne \frac{1}{2}$ cannot attain the maximum.

