Exercise 1 of February 23, 2017

http://www.isi.ee.ethz.ch/teaching/courses/it2.html

Problem 1 — *Differential Entropy*

Evaluate the differential entropy $h(X) = -\int_{-\infty}^{\infty} f_X(x) \log f_X(x)\,\mathrm{d}x$ in the following situations:

a) the exponential distribution, $f_X(x) = \lambda e^{-\lambda x}\,\mathrm{I}\{x \geq 0\}$;

b) the Laplace distribution, $f_X(x) = \frac{\lambda}{2}\,e^{-\lambda|x|}$; and

c) the sum of X_1 and X_2, where X_1 and X_2 are independent Gaussian random variables with means μ_i and variances σ_i^2, $i = 1, 2$.

Problem 2 — *Scaling Property*

Let $\mathbf{X}$ be a random n-vector of density $f_{\mathbf{X}}(\cdot)$ and finite differential entropy

$$h(\mathbf{X}) = -\int_{\mathbb{R}^n} f_{\mathbf{X}}(\mathbf{x}) \log f_{\mathbf{X}}(\mathbf{x})\,\mathrm{d}\mathbf{x},$$

and let A be a deterministic $n \times n$ nonsingular matrix. Show that

$$h(\mathsf{A}\mathbf{X}) = \log|\det \mathsf{A}\,| + h(\mathbf{X}).$$

Problem 3 — *AEP for Gaussian Random Variables*

a) Let $X_1, \ldots, X_n$ be a sequence of IID Gaussian random variables X of mean μ and variance σ^2. Find the probability density of the random variable

$$Y = -\frac{1}{n} \log f_{X_1,\ldots,X_n}(X_1, \ldots, X_n) - h(X),$$

where $h(X)$ is the differential entropy of X. Does this distribution depend on σ^2 and μ?

b) Let $(X_1, \ldots, X_n)^{\mathsf{T}}$ be jointly Gaussian with mean vector $\boldsymbol{\mu}$ and $n \times n$ covariance matrix K. Find the probability density of the random variable

$$Y = -\frac{1}{n} \log f_{X_1,\ldots,X_n}(X_1, \ldots, X_n) - \frac{1}{n} h(X_1, \ldots, X_n).$$

c) Show that if $(X_1, \ldots, X_n)^\mathsf{T}$ is a random n-vector with density $f_{X_1,\ldots,X_n}(\cdot)$, and

$$(Y_1, \ldots, Y_n)^\mathsf{T} = \mathsf{A}(X_1, \ldots, X_n)^\mathsf{T},$$

where A is a nonsingular $n \times n$ matrix, then

$$-\log f_{Y_1,\ldots,Y_n}(Y_1, \ldots, Y_n) - h(Y_1, \ldots, Y_n) = -\log f_{X_1,\ldots,X_n}(X_1, \ldots, X_n) - h(X_1, \ldots, X_n).$$

Hint: Note that for Z_i IID $\sim \mathcal{N}(0,1)$, the random variable

$$\chi_n^2 = \sum_{i=1}^{n} Z_i^2$$

is central chi-square distributed with n degrees of freedom and has probability density function

$$f_{\chi_n^2}(x) = \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}\, x^{\frac{n}{2}-1}\, e^{-\frac{x}{2}}\, \mathrm{I}\{x > 0\},$$

where the indicator function $\mathrm{I}\{x > 0\} = 1$ if $x > 0$, and $\mathrm{I}\{x > 0\} = 0$ if $x \leq 0$.