Exercise 2 of March 2, 2017

http://www.isi.ee.ethz.ch/teaching/courses/it2.html

Problem 1 — *Log-Concavity of Determinants*

Let K_1 and K_2 be two symmetric positive semidefinite $n \times n$ matrices. Prove the result of Ky Fan[1]:

$$\det(\lambda \mathsf{K}_1 + \bar{\lambda}\mathsf{K}_2) \geq \det(\mathsf{K}_1)^{\lambda} \det(\mathsf{K}_2)^{\bar{\lambda}}, \quad \text{for } 0 \leq \lambda \leq 1,\ \bar{\lambda} = 1 - \lambda,$$

where $\det(\mathsf{K})$ denotes the determinant of K.
Hint: Let $\mathbf{Z} = \mathbf{X}_\theta$, *where* $\mathbf{X}_1 \sim \mathcal{N}(\mathbf{0}, \mathsf{K}_1)$, $\mathbf{X}_2 \sim \mathcal{N}(\mathbf{0}, \mathsf{K}_2)$ *and* $\theta = \text{Bernoulli}(\lambda)$*. Then use* $h(\mathbf{Z}|\theta) \leq h(\mathbf{Z})$*.*

Problem 2 — *Continuous Fano Inequality*

Here is the estimation counterpart to Fano's inequality. Let X be a random variable with differential entropy $h(X)$ measured in bits. Let $\hat{X}$ be an estimate of X, and let $\tau^2 = \mathsf{E}\big[(X - \hat{X})^2\big]$ be the expected prediction error.

a) Let $\hat{X}$ be independent of X. Show that

$$\tau^2 \geq \frac{1}{2\pi e} 2^{2h(X)}.$$

When does equality hold?

b) Allow the estimate $\hat{X}$ to depend on some given side information Y. Show that

$$\tau^2 \geq \frac{1}{2\pi e} 2^{2h(X|Y)}.$$

Problem 3 — *Maximum Differential Entropy*

Find the maximum differential entropy density $f(\cdot)$ among all nonnegative random variables that satisfy $\mathsf{E}[X] = \alpha_1$ and $\mathsf{E}[\ln X] = \alpha_2$. That is, maximize $h(X)$ subject to

$$f(x) = 0 \quad \forall x < 0,$$
$$\int_0^\infty f(x)\, x \,\mathrm{d}x = \alpha_1,$$
$$\int_0^\infty f(x) \log x \,\mathrm{d}x = \alpha_2.$$

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[1] Ky Fan. On a Theorem of Weyl concerning the eigenvalues of linear transformations II. *Proc. National Accad. Sci. U.S.*, 36: 31–35, 1950.

Problem 4 — *Minimum $D(P\|Q)$ under Constraints on P*

We wish to find the (parametric form) of the probability mass function $P(x)$, $x \in \{1, 2, \dots\}$ that minimizes the relative entropy $D(P\|Q)$ over all P such that $\sum_x P(x) g_i(x) = \alpha_i$, $i = 1, 2, \dots$.

a) Use Lagrange multipliers to guess that

$$P^*(x) = Q(x)\, e^{\lambda_0 + \sum_{i=1}^{\infty} \lambda_i g_i(x)}$$

achieves this minimum if there exist λ_i's satisfying the α_i constraints. This generalizes the theorem on maximum entropy distributions subject to constraints.

b) Verify that P^* indeed minimizes $D(P\|Q)$.

Problem 5 — *Maximum Conditional Differential Entropy*

Let the random variables X and Y be of zero mean, variance σ_X^2 and σ_Y^2, respectively, and covariance $\rho\sigma_X\sigma_Y$. Show that

$$h(X|Y) \leq \frac{1}{2} \log\Bigl(2\pi e\bigl(1 - \rho^2\bigr)\sigma_X^2\Bigr),$$

and that equality is achieved when X and Y are jointly Gaussian.
Hint: Express $h(X|Y)$ as $h(X - \alpha Y|Y)$ for some constant α.