Information Theory II

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http://www.isi.ee.ethz.ch/teaching/courses/it2.html

Problem 1 — *Gaussian Multipath Channel*

Consider a channel with an average-power constraint E_s where the signal takes two different paths and the two noisy signals are added together at the receiver antenna. More explicitly, the output of the channel is given by

$$Y = Y_1 + Y_2,$$

where

$$Y_1 = X + Z_1,$$
$$Y_2 = X + Z_2.$$

a) Find the capacity of this channel if Z_1 and Z_2 are jointly Gaussian with covariance matrix

$$\mathsf{K}_{\mathbf{ZZ}} = \begin{pmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{pmatrix}.$$

b) What is the capacity for $\rho = 0$, $\rho = 1$, and $\rho = -1$?

Problem 2 — *Additive Noise Channel*

Consider a power limited memoryless additive noise channel where the output Y_k is equal to the sum of the input signal x_k and the noise sample Z_k:

$$Y_k = x_k + Z_k.$$

The transmitter is average-power limited, i.e., for every message m, it is required that

$$\frac{1}{n}\sum_{k=1}^{n} x_k^2(m) \le \mathsf{E}_\mathrm{s}.$$

The IID noise process $\{Z_k\}$ is independent of the channel input ($\{Z_k\} \perp\!\!\!\perp \{X_k\}$) and equal to 0 with probability $1/10$ and is otherwise Gaussian with zero mean and variance $\sigma^2 > 0$:

$$Z_k = \begin{cases} 0 & \text{with probability } \frac{1}{10}, \\ G_k & \text{with probability } \frac{9}{10}, \end{cases}$$

where $\{G_k\}$ is IID $\sim \mathcal{N}(0, \sigma^2)$. What is the capacity of this channel and how can it be achieved?

Problem 3 ***DMC with a Cost Constraint***

Consider a discrete memoryless channel with a finite input alphabet $\mathcal{X}$. For every input symbol $x \in \mathcal{X}$, define a cost $b(x) \in \mathbb{R}_0^+$. A cost constraint on a codebook for this channel is

$$\frac{1}{n}\sum_{i=1}^{n} b\bigl(x_i(m)\bigr) \leq \beta, \quad m \in \bigl\{1, \ldots, 2^{n\mathsf{R}}\bigr\}, \tag{1}$$

where β is some positive constant. A rate R is said to be achievable on this channel under cost constraint β if there exists a sequence of (n, R) codebooks satisfying (1) whose maximum error probability tends to zero as n tends to infinity. The capacity under cost constraint β, denoted by $\mathsf{C}(\beta)$, is defined as the supremum over all rates that are achievable under cost constraint β.
Show that

$$\mathsf{C}(\beta) = \sup_{\mathsf{E}[b(X)] \leq \beta} I(X;Y). \tag{2}$$

Further, show that (2) still holds if we replace (1) by the weaker condition

$$\frac{1}{2^{n\mathsf{R}}}\sum_{m=1}^{2^{n\mathsf{R}}} \frac{1}{n}\sum_{i=1}^{n} b\bigl(x_i(m)\bigr) \leq \beta.$$

