Information Theory II

Spring Semester 2017

Prof. Dr. A. Lapidoth / Prof. Dr. S. Moser

Signal and Information Processing Laboratory

Institut für Signal- und Informationsverarbeitung

Exercise 5 of March 23, 2017

http://www.isi.ee.ethz.ch/teaching/courses/it2.html

Problem 1 — *Waterfilling Example*

Let n be a positive integer and let $a_1, a_2, \ldots, a_n$ be positive reals. Which choice of $b_1, b_2, \ldots, b_n$ maximizes $\sum_{i=1}^{n} \sqrt{a_i + b_i}$ subject to $b_i \geq 0$ for $i = 1, \ldots, n$ and $\sum_{i=1}^{n} b_i = 1$?

Problem 2 — *Parallel Channels and Waterfilling*

Consider a pair of parallel Gaussian channels, i.e.,

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix},$$

where

$$\mathbf{Z} = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim \mathcal{N}\left(\mathbf{0}, \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}\right)$$

and every codeword has to satisfy the total average-power constraint

$$\frac{1}{n} \sum_{i=1}^{n} \|\mathbf{X}_i\|^2 = \frac{1}{n} \sum_{i=1}^{n} \left(X_{1,i}^2 + X_{2,i}^2\right) \leq \mathsf{E}.$$

Assume $\sigma_1^2 > \sigma_2^2$. At what power does the channel stop behaving like a single channel with noise variance σ_2^2 and begin behaving like a pair of channels?

Problem 3 — *Dependent Channels and Waterfilling*

Consider a pair of parallel Gaussian channels, i.e.,

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix},$$

where

$$\mathbf{Z} = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim \mathcal{N}\left(\mathbf{0}, \begin{pmatrix} 8 & 2 \\ 2 & 5 \end{pmatrix}\right)$$

and every codeword has to satisfy the total average-power constraint

$$\frac{1}{n} \sum_{i=1}^{n} \|\mathbf{X}_i\|^2 = \frac{1}{n} \sum_{i=1}^{n} \left(X_{1,i}^2 + X_{2,i}^2\right) \leq \mathsf{E}.$$

a) Assume that E is very small. Does an optimal transmission scheme put all power into one channel? If yes, into which one? If no, why not?

b) What is the capacity of the parallel channel?

c) How can the capacity of the parallel channel be achieved?