Information Theory II

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http://www.isi.ee.ethz.ch/teaching/courses/it2.html

Problem 1 ***Capacity of Two MACs***

Determine the capacity region of the following multiple-access channels:

a) Additive modulo-2 MAC: $X_1, X_2 \in \{0, 1\}$, $Y = X_1 \oplus X_2$.

b) Multiplicative MAC: $X_1, X_2 \in \{-1, 1\}$, $Y = X_1 \cdot X_2$.

Problem 2 ***Cooperative Capacity of a MAC***

a) Consider a discrete MAC with two users. What is the capacity region if both transmitters have access to both messages $M_1 \in \{1, \ldots, 2^{n\mathsf{R}_1}\}$ and $M_2 \in \{1, \ldots, 2^{n\mathsf{R}_2}\}$?

b) Evaluate this region for the binary erasure MAC

$$Y = X_1 + X_2,$$

where $X_1 \in \{0, 1\}$, $X_2 \in \{0, 1\}$, and the addition is the standard addition in $\mathbb{Z}$. Compare with the case where each transmitter only has access to its own message.

Problem 3 ***Necessity of Time-Sharing on the MAC***

Consider a multiple-access channel with binary alphabets $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{Y} = \{0, 1\}$. The channel law is described as follows:

- if $(X_1, X_2) = (0, 0)$, then $Y = 0$;
- if $(X_1, X_2) = (0, 1)$, then $Y = 1$;
- if $(X_1, X_2) = (1, 0)$, then $Y = 1$;
- if $(X_1, X_2) = (1, 1)$, then Y takes on the values 0 and 1 equiprobably.

a) Show that the rate pairs $(0, 1)$ bits and $(1, 0)$ bits are achievable.

b) Show that for any product distribution $P_{X_1} \cdot P_{X_2}$ with $P_{X_1}(1)\, P_{X_2}(1) > 0$,

$$I(X_1, X_2; Y) < 1 \text{ bit}.$$

c) Argue that there are points in the capacity region of this MAC that can only be achieved by time-sharing; that is, there exist rate pairs $(\mathsf{R}_1, \mathsf{R}_2)$ that lie in the capacity region for the channel, but not in the region defined by

$$\begin{aligned} \mathsf{R}_1 &\leq I\big(X_1; Y \,\big|\, X_2\big), \\ \mathsf{R}_2 &\leq I\big(X_2; Y \,\big|\, X_1\big), \\ \mathsf{R}_1 + \mathsf{R}_2 &\leq I\big(X_1, X_2; Y\big) \end{aligned}$$

for any product distribution $P_{X_1} P_{X_2}$. Hence time-sharing strictly enlarges the capacity region.

Problem 4 — *The MAC with Dependent Nonuniform Messages*

Given a MAC, some $\epsilon > 0$, and a sequence of priors on the messages

$$\pi^{(n)}_{m_1,m_2} = \Pr[M_1 = m_1, M_2 = m_2], \quad (m_1, m_2) \in \{1, \ldots, 2^{n\mathsf{R}_1}\} \times \{1, \ldots, 2^{n\mathsf{R}_2}\}, \quad n = 1, 2, \ldots,$$

not necessarily uniform and not necessarily such that M_1 and M_2 are independent, show that if for some product distribution $P_{X_1} P_{X_2}$ the rates R_1 and R_2 satisfy

$$\begin{aligned} \mathsf{R}_1 &< I(X_1; Y|X_2), \\ \mathsf{R}_2 &< I(X_2; Y|X_1), \\ \mathsf{R}_1 + \mathsf{R}_2 &< I(X_1, X_2; Y), \end{aligned}$$

then, for all sufficiently large blocklengths n, there exist a rate-R_1 blocklength-n codebook $\mathcal{C}_1$, a rate-R_2 blocklength-n codebook $\mathcal{C}_2$, and a decoding rule ϕ such that

$$\sum_{m_1,m_2} \pi^{(n)}_{m_1,m_2} \Pr(\text{error}|M_1 = m_1, M_2 = m_2) \leq \epsilon.$$