Exercise 6 of March 30, 2017

Problem 1

Capacity of Two MACs

Determine the capacity region of the following multiple-access channels:

a) Additive modulo-2 MAC: $X_1, X_2 \in \{0, 1\}, Y = X_1 \oplus X_2$.

b) Multiplicative MAC: $X_1, X_2 \in \{-1, 1\}, Y = X_1 \cdot X_2$.

Problem 2

Cooperative Capacity of a MAC

a) Consider a discrete MAC with two users. What is the capacity region if both transmitters have access to both messages $M_1 \in \{1, \ldots, 2^{nR_1}\}$ and $M_2 \in \{1, \ldots, 2^{nR_2}\}$?

b) Evaluate this region for the binary erasure MAC

$Y = X_1 + X_2$,

where $X_1 \in \{0, 1\}, X_2 \in \{0, 1\}$, and the addition is the standard addition in $\mathbb{Z}$. Compare with the case where each transmitter only has access to its own message.

Problem 3

Necessity of Time-Sharing on the MAC

Consider a multiple-access channel with binary alphabets $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{Y} = \{0, 1\}$. The channel law is described as follows:

- if $(X_1, X_2) = (0, 0)$, then $Y = 0$;
- if $(X_1, X_2) = (0, 1)$, then $Y = 1$;
- if $(X_1, X_2) = (1, 0)$, then $Y = 1$;
- if $(X_1, X_2) = (1, 1)$, then $Y$ takes on the values 0 and 1 equiprobably.

a) Show that the rate pairs $(0, 1)$ bits and $(1, 0)$ bits are achievable.

b) Show that for any product distribution $P_{X_1} \cdot P_{X_2}$ with $P_{X_1}(1) P_{X_2}(1) > 0$,

$I(X_1, X_2; Y) < 1$ bit.
c) Argue that there are points in the capacity region of this MAC that can only be achieved by time-sharing; that is, there exist rate pairs \((R_1, R_2)\) that lie in the capacity region for the channel, but not in the region defined by

\[
\begin{align*}
R_1 &\leq I(X_1; Y | X_2), \\
R_2 &\leq I(X_2; Y | X_1), \\
R_1 + R_2 &\leq I(X_1, X_2; Y)
\end{align*}
\]

for any product distribution \(P_{X_1} P_{X_2}\). Hence time-sharing strictly enlarges the capacity region.

**Problem 4**

*The MAC with Dependent Nonuniform Messages*

Given a MAC, some \(\epsilon > 0\), and a sequence of priors on the messages

\[
\pi_{m_1, m_2}^{(n)} = \Pr[M_1 = m_1, M_2 = m_2], \quad (m_1, m_2) \in \{1, \ldots, 2^{nR_1}\} \times \{1, \ldots, 2^{nR_2}\}, \quad n = 1, 2, \ldots,
\]

not necessarily uniform and not necessarily such that \(M_1\) and \(M_2\) are independent, show that if for some product distribution \(P_{X_1} P_{X_2}\) the rates \(R_1\) and \(R_2\) satisfy

\[
\begin{align*}
R_1 &< I(X_1; Y | X_2), \\
R_2 &< I(X_2; Y | X_1), \\
R_1 + R_2 &< I(X_1, X_2; Y),
\end{align*}
\]

then, for all sufficiently large blocklengths \(n\), there exist a rate-\(R_1\) blocklength-\(n\) codebook \(C_1\), a rate-\(R_2\) blocklength-\(n\) codebook \(C_2\), and a decoding rule \(\phi\) such that

\[
\sum_{m_1, m_2} \pi_{m_1, m_2}^{(n)} \Pr(\text{error} | M_1 = m_1, M_2 = m_2) \leq \epsilon.
\]

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