Information Theory II

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Model Answers to Exercise 6 of March 30, 2017

http://www.isi.ee.ethz.ch/teaching/courses/it2.html

Problem 1 ***Capacity of Two MACs***

a) The rate pair $(\mathsf{R}_1, \mathsf{R}_2) = (1, 0)$ bits is achievable: if X_2 is always zero, then $Y = X_1$, so the first user can transmit one bit per channel use with zero probability of error. By symmetry, the rate pair $(\mathsf{R}_1, \mathsf{R}_2) = (0, 1)$ bits is also achievable. Because the capacity region is convex, all rate pairs $(\mathsf{R}_1, \mathsf{R}_2)$ satisfying $\mathsf{R}_1 + \mathsf{R}_2 \leq 1$ bit are in the capacity region.

The capacity region does not contain any rate pair $(\mathsf{R}_1, \mathsf{R}_2)$ with $\mathsf{R}_1 + \mathsf{R}_2 > 1$ bit because

$$I(X_1, X_2; Y) = H(Y) - H(Y|X_1, X_2) \leq H(Y) \leq 1 \text{ bit},$$

so every pentagon region only contains points satisfying $\mathsf{R}_1 + \mathsf{R}_2 \leq 1$ bit, and thus every point in the convex hull of the union of all pentagon regions has to satisfy $\mathsf{R}_1 + \mathsf{R}_2 \leq 1$ bit, too.

Therefore, the capacity region is the set of all rate pairs $(\mathsf{R}_1, \mathsf{R}_2)$ satisfying $\mathsf{R}_1 + \mathsf{R}_2 \leq 1$ bit. It is depicted in Figure 1.

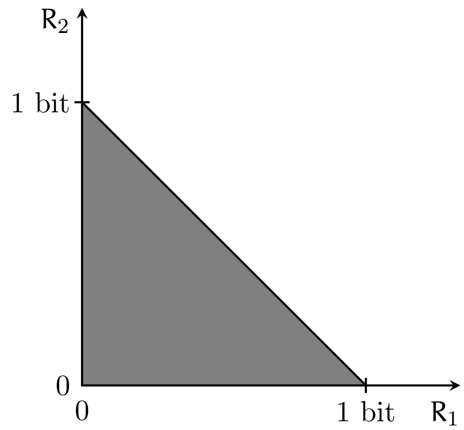

Figure 1: Capacity region of the MAC.

b) This channel is equivalent to the channel from Part a), so it has the same capacity region. To see the equivalence, map 1 to 0, map -1 to 1, and change the multiplication to modulo-2 addition.

a) If both transmitters have access to both messages, the channel is equivalent to a single-user channel with input alphabet $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$ and message set $\{1, \ldots, 2^{n\mathsf{R}_1}\} \times \{1, \ldots, 2^{n\mathsf{R}_2}\}$. The capacity of the single-user channel is

$$\mathsf{C} = \sup_{P_X} I(X;Y) = \sup_{P_{X_1X_2}} I(X_1, X_2; Y).$$

Any combination of rate pairs $(\mathsf{R}_1, \mathsf{R}_2)$ is possible on the MAC as long as the sum rate does not exceed C, i.e., the capacity region is the set of all rate pairs $(\mathsf{R}_1, \mathsf{R}_2)$ satisfying

$$\mathsf{R}_1 + \mathsf{R}_2 \leq \mathsf{C}.$$

b) The output alphabet has cardinality three, so C is upperbounded by $\log 3$ because

$$I(X_1, X_2; Y) = H(Y) - H(Y|X_1, X_2) \leq H(Y) \leq \log 3 \approx 1.58 \text{ bits}$$

for all input distributions. This bound can be achieved for example with the input distribution

$P_{X_1X_2}(x_1, x_2)$	$x_2 = 0$	$x_2 = 1$
$x_1 = 0$	$1/3$	$1/3$
$x_1 = 1$	0	$1/3.$

The noncooperative capacity region has been derived in the lecture; it consists of all rate pairs $(\mathsf{R}_1, \mathsf{R}_2)$ satisfying

$$\begin{aligned} \mathsf{R}_1 &\leq 1 \text{ bit}, \\ \mathsf{R}_2 &\leq 1 \text{ bit}, \\ \mathsf{R}_1 + \mathsf{R}_2 &\leq 1.5 \text{ bits}. \end{aligned}$$

In Figure 2, the cooperative and the noncooperative capacity region are shown. Note that cooperation is not only beneficial in the corners where the rate of one user exceeds 1 bit, but it also improves the sum rate because the optimization of the mutual information $I(X_1, X_2; Y)$ is over all joint distributions $P_{X_1X_2}$ and not just over product distributions $P_{X_1} \cdot P_{X_2}$.

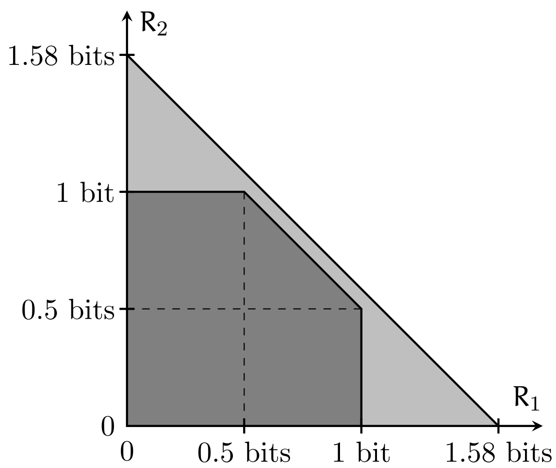

Figure 2: Cooperative (light gray) and noncooperative (gray) capacity region of the binary erasure MAC.

a) If we deterministically set $X_1 = 0$, then $Y = X_2$. The binary input from the second user can be recovered perfectly and hence the rate pair $(0, 1)$ bits is achievable. By the symmetry of the channel, the rate pair $(1, 0)$ bits is also achievable.

b) Measuring mutual information in bits, we have for any product distribution $P_{X_1} \cdot P_{X_2}$

$$\begin{aligned} I(X_1, X_2; Y) &= H(Y) - H(Y|X_1, X_2) \\ &= H(Y) - \sum_{x_1, x_2} P_{X_1 X_2}(x_1, x_2) H(Y|X_1 = x_1, X_2 = x_2) \\ &\overset{\text{(i)}}{=} H(Y) - \sum_{x_1, x_2} P_{X_1}(x_1) P_{X_2}(x_2) H(Y|X_1 = x_1, X_2 = x_2) \\ &\overset{\text{(ii)}}{=} H(Y) - P_{X_1}(1) P_{X_2}(1) \\ &\leq 1 - P_{X_1}(1) P_{X_2}(1), \end{aligned} \tag{1}$$

where (i) holds because we restrict ourselves to product distributions $P_{X_1} \cdot P_{X_2}$; and (ii) holds because $H(Y|X_1 = x_1, X_2 = x_2)$ is equal to 1 bit if $(x_1, x_2) = (1, 1)$ and zero otherwise. If $P_{X_1}(1) P_{X_2}(1) > 0$, then (1) implies that $I(X_1, X_2; Y) < 1$ bit.

c) From Part a) we know that the rate pairs $(0, 1)$ bits and $(1, 0)$ bits are in the capacity region, so by the convexity of the capacity region, all rate pairs $(\mathsf{R}_1, \mathsf{R}_2)$ satisfying $\mathsf{R}_1 + \mathsf{R}_2 \leq 1$ bit are in the capacity region. In particular, the rate pair $(0.5, 0.5)$ bits is in the capacity region. On the other hand, the rate pair $(0.5, 0.5)$ bits is not in any pentagon region: From Part b) we know that this could only be possible if $P_{X_1}(1) P_{X_2}(1) = 0$ would hold. But if $P_{X_1}(1) = 0$ would hold, then $0.5 = \mathsf{R}_1 \leq I(X_1; Y|X_2) = 0$, which is a contradiction; similarly, $P_{X_2}(1) = 0$ is impossible. Hence, the rate pair $(0.5, 0.5)$ bits cannot be in any pentagon region.

Unfortunately, this argument is not sufficient to show that time-sharing is necessary: it could be possible that rate pairs arbitrarily close to $(0.5, 0.5)$ bits are in a pentagon region; this would still be okay, since we do not care about what happens for rate pairs on the boundary of the capacity region. For the interested reader, we now prove by contradiction that the rate pair $\bigl(\frac{1}{2} - \delta, \frac{1}{2} - \delta\bigr)$, which lies in the interior of the capacity region for $\delta > 0$, is not in any pentagon region if δ is small enough, thus it cannot be achieved without time-sharing. To that end, assume that the rate pair $\bigl(\frac{1}{2} - \delta, \frac{1}{2} - \delta\bigr)$ is in a pentagon region, i.e., assume that there exists a product distribution $P_{X_1} \cdot P_{X_2}$ with

$$\tfrac{1}{2} - \delta \leq I(X_1; Y|X_2), \tag{2}$$

$$\tfrac{1}{2} - \delta \leq I(X_2; Y|X_1), \tag{3}$$

$$\bigl(\tfrac{1}{2} - \delta\bigr) + \bigl(\tfrac{1}{2} - \delta\bigr) \leq I(X_1, X_2; Y). \tag{4}$$

From (2) we see that

$$\frac{1}{2} - \delta \leq I(X_1; Y|X_2) = H(X_1|X_2) - H(X_1|X_2, Y) \leq H(X_1|X_2) = H(X_1) = H_{\mathrm{b}}\bigl(P_{X_1}(1)\bigr),$$

which is equivalent to $P_{X_1}(1) \geq H_{\mathrm{b}}^{-1}\bigl(\frac{1}{2} - \delta\bigr)$. Similarly, $P_{X_2}(1) \geq H_{\mathrm{b}}^{-1}\bigl(\frac{1}{2} - \delta\bigr)$ follows from (3). From Part b) we know that (4) implies

$$P_{X_1}(1) P_{X_2}(1) \leq 2\delta.$$

Combining these results, we obtain

$$\left[H_{\mathrm{b}}^{-1}\left(\frac{1}{2} - \delta\right)\right]^2 \leq P_{X_1}(1) P_{X_2}(1) \leq 2\delta \tag{5}$$

and arrive at the desired contradiction: (5) cannot hold for arbitrarily small values of δ since the LHS tends to $\left[H_{\mathrm{b}}^{-1}\left(\frac{1}{2}\right)\right]^2 > 0$ as δ tends to zero, while the RHS tends to zero as δ tends to zero. Consequently, no product distribution $P_{X_1} \cdot P_{X_2}$ can satisfy (2)–(4) if δ is small enough.

Problem 4 — *The MAC with Dependent Nonuniform Messages*

We generate a rate-R_1 blocklength-n codebook $\mathcal{C}_1$ for the first user by drawing IID from P_{X_1}, and we generate a rate-R_2 blocklength-n codebook $\mathcal{C}_2$ for the second user by drawing IID from P_{X_2}. In the same way as in the case where the messages are uniform and independent, we can show that

$$\sum_{\mathcal{C}_1,\mathcal{C}_2} \Pr(\mathcal{C}_1,\mathcal{C}_2) \Pr(\text{error}|M_1 = 1, M_2 = 1) \le \epsilon \tag{6}$$

holds for all sufficiently large n. Therefore, we have for n large enough

$$\begin{aligned}
&\sum_{\mathcal{C}_1,\mathcal{C}_2} \Pr(\mathcal{C}_1,\mathcal{C}_2) \sum_{m_1,m_2} \pi^{(n)}_{m_1,m_2} \Pr(\text{error}|M_1 = m_1, M_2 = m_2) \\
&\quad = \sum_{m_1,m_2} \pi^{(n)}_{m_1,m_2} \sum_{\mathcal{C}_1,\mathcal{C}_2} \Pr(\mathcal{C}_1,\mathcal{C}_2) \Pr(\text{error}|M_1 = m_1, M_2 = m_2) \\
&\quad \overset{\text{(i)}}{=} \sum_{m_1,m_2} \pi^{(n)}_{m_1,m_2} \sum_{\mathcal{C}_1,\mathcal{C}_2} \Pr(\mathcal{C}_1,\mathcal{C}_2) \Pr(\text{error}|M_1 = 1, M_2 = 1) \\
&\quad \overset{\text{(ii)}}{\le} \epsilon,
\end{aligned}$$

where (i) follows from the symmetry of the code construction; and (ii) follows from (6). Since the probability of error averaged over all codebook realizations is less than or equal to ϵ, there must exist a pair of codebooks $(\mathcal{C}_1,\mathcal{C}_2)$ with

$$\sum_{m_1,m_2} \pi^{(n)}_{m_1,m_2} \Pr(\text{error}|M_1 = m_1, M_2 = m_2) \le \epsilon.$$