Exercise 8 of April 13, 2017

http://www.isi.ee.ethz.ch/teaching/courses/it2.html

Problem 1 — *Capacity Region of Broadcast Channels Depends Only on Conditional Marginals*

Show that the capacity region of broadcast channels depends only on $P_{Y_1|X}$ and $P_{Y_2|X}$. To that end, define for a broadcast channel with encoder $f\colon \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{X}^n$ and decoders $\phi_1\colon \mathcal{Y}_1^n \to \mathcal{M}_1$ and $\phi_2\colon \mathcal{Y}_2^n \to \mathcal{M}_2$

$$\begin{aligned} P_1 &\triangleq \Pr\bigl[\phi_1(Y_1^n) \neq M_1\bigr], \\ P_2 &\triangleq \Pr\bigl[\phi_2(Y_2^n) \neq M_2\bigr], \\ P_{\mathrm{e}} &\triangleq \Pr\bigl[(\phi_1(Y_1^n), \phi_2(Y_2^n)) \neq (M_1, M_2)\bigr]. \end{aligned}$$

Show that

$$\max\{P_1, P_2\} \leq P_{\mathrm{e}} \leq 2 \cdot \max\{P_1, P_2\}$$

and use this result to prove the claim.

Remark: Note that the probability of error P_{e} depends not only on the conditional marginals $P_{Y_1|X}$ and $P_{Y_2|X}$, but also on the conditional joint distribution $P_{Y_1,Y_2|X}$. But whether P_{e} can be driven to zero at rates $(\mathsf{R}_1, \mathsf{R}_2)$ or not depends only on the conditional marginals $P_{Y_1|X}$ and $P_{Y_2|X}$.

Problem 2 — *Capacity Points*

For a degraded broadcast channel $X \multimap\!\!- Y_1 \multimap\!\!- Y_2$, define

$$\begin{aligned} \mathsf{R}_1^* &\triangleq \sup\{\mathsf{R}_1\colon (\mathsf{R}_1, 0) \text{ is achievable}\}, \\ \mathsf{R}_2^* &\triangleq \sup\{\mathsf{R}_2\colon (0, \mathsf{R}_2) \text{ is achievable}\}. \end{aligned}$$

a) Determine R_1^* and R_2^*.

b) Show that $\mathsf{R}_1^* \geq \mathsf{R}_2^*$.

Problem 3 — *Broadcast BEC*

Consider a broadcast channel whose input X takes values in $\{0, 1\}$, and whose outputs Y_1 and Y_2 take values in $\{0, 1, ?\}$. The channel from X to Y_1 is a binary erasure channel with erasure probability α_1, and the channel from X to Y_2 is a binary erasure channel with erasure probability α_2, where $\alpha_1 < \alpha_2$. What is the capacity region of this channel?

Problem 4 *Converse for Deterministic Broadcast Channel*

Consider a deterministic broadcast channel where the two outputs $Y_1 = f_1(X)$ and $Y_2 = f_2(X)$ are both deterministic functions of the input X. Assume that a rate pair $(\mathsf{R}_1, \mathsf{R}_2)$ is achievable on this channel. Show that there exists a distribution P_X on X such that, for the joint distribution P_{Y_1,Y_2} induced by P_X and by the channel law,

$$\begin{aligned} \mathsf{R}_1 &\leq H(Y_1), \\ \mathsf{R}_2 &\leq H(Y_2), \\ \mathsf{R}_1 + \mathsf{R}_2 &\leq H(Y_1, Y_2). \end{aligned}$$