Information Theory II

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Prof. Dr. A. Lapidoth / Prof. Dr. S. Moser

Signal and Information Processing Laboratory
Institut für Signal- und Informationsverarbeitung

Model Answers to Exercise 9 of April 27, 2017

http://www.isi.ee.ethz.ch/teaching/courses/it2.html

Problem 1 — *Feedback Capacity of a Channel with Causal State Information at the Transmitter*

The proof that feedback does not increase the capacity in this setting follows the same lines as the proof of the converse for the case without feedback. In particular, we have

$$\begin{aligned}
I\bigl(M;\hat{M}\bigr) &\le I\bigl(M;Y^n\bigr) \\
&= H\bigl(Y^n\bigr) - H\bigl(Y^n\big|M\bigr) \\
&= \sum_{i=1}^{n}\Bigl[H\bigl(Y_i\big|Y^{i-1}\bigr) - H\bigl(Y_i\big|M,Y^{i-1}\bigr)\Bigr] \\
&\le \sum_{i=1}^{n}\Bigl[H\bigl(Y_i\bigr) - H\bigl(Y_i\big|M,Y^{i-1},S^{i-1}\bigr)\Bigr] \\
&\stackrel{\text{(i)}}{=} \sum_{i=1}^{n}\Bigl[H(Y_i) - H(Y_i|U_i)\Bigr] \\
&= \sum_{i=1}^{n} I(U_i;Y_i) \\
&\stackrel{\text{(ii)}}{\le} n\cdot \max_{P_U,\, f\colon \mathcal{U}\times\mathcal{S}\to\mathcal{X}} I(U;Y),
\end{aligned}$$

where in (i) we identified the auxiliary random variable $U_i \triangleq \bigl(M, Y^{i-1}, S^{i-1}\bigr)$; and (ii) holds because X_i is a function of (U_i, S_i) and because U_i is independent of S_i. As in the case without feedback, the proof of the converse is finished by applying Fano's inequality.

Problem 2 — *A DMC Where the Transmitter Knows Only the Previous States*

We have

$$\begin{aligned}
I\bigl(M;\hat{M}\bigr) &\le I\bigl(M;Y^n\bigr) \\
&= H\bigl(Y^n\bigr) - H\bigl(Y^n\big|M\bigr) \\
&= \sum_{i=1}^{n}\Bigl[H\bigl(Y_i\big|Y^{i-1}\bigr) - H\bigl(Y_i\big|M,Y^{i-1}\bigr)\Bigr] \\
&\le \sum_{i=1}^{n}\Bigl[H\bigl(Y_i\bigr) - H\bigl(Y_i\big|M,Y^{i-1},S^{i-1}\bigr)\Bigr]
\end{aligned}$$

$$\begin{aligned}
&\stackrel{\text{(i)}}{=} \sum_{i=1}^{n} \Bigl[H(Y_i) - H\bigl(Y_i \big| M, Y^{i-1}, S^{i-1}, X_i\bigr) \Bigr] \\
&\stackrel{\text{(ii)}}{=} \sum_{i=1}^{n} \Bigl[H(Y_i) - H\bigl(Y_i \big| X_i\bigr) \Bigr] \\
&= \sum_{i=1}^{n} I(X_i; Y_i) \\
&\le n \cdot \max_{P_X} I(X;Y),
\end{aligned}$$

where (i) holds because X_i is a function of $\bigl(M, S^{i-1}\bigr)$; and (ii) holds since the channel is memoryless and thus $\bigl(M, Y^{i-1}, S^{i-1}\bigr) \multimap X_i \multimap Y_i$ form a Markov chain. Applying Fano's inequality shows that the capacity is upperbounded by

$$\max_{P_X} I(X;Y), \tag{1}$$

and any rate smaller than (1) can be achieved by ignoring the state information, so the capacity of this channel is given by (1).

Problem 3 — *A BSC With a State*

When the state is known causally at the transmitter and at the receiver, the capacity is

$$\max_{P_{X|S}} I(X;Y|S).$$

Conditional on $S = 0$ or $S = 1$, the channel is a BSC with crossover probability ϵ. Thus,

$$\begin{aligned}
\max_{P_{X|S}} I(X;Y|S) &= \Pr[S=0] \cdot \max_{P_{X|S=0}} I(X;Y|S=0) + \Pr[S=1] \cdot \max_{P_{X|S=1}} I(X;Y|S=1) \\
&= \Pr[S=0] \cdot \bigl(1 - H_{\text{b}}(\epsilon)\bigr) + \Pr[S=1] \cdot \bigl(1 - H_{\text{b}}(\epsilon)\bigr) \\
&= 1 - H_{\text{b}}(\epsilon),
\end{aligned}$$

which implies that the capacity is upperbounded by $1 - H_{\text{b}}(\epsilon)$ if the state is known causally only at the transmitter. We now argue that $1 - H_{\text{b}}(\epsilon)$ is indeed the capacity in this case: the encoder can simply subtract S from the input, so the input-output relation becomes

$$Y = x \oplus Z,$$

which is the channel law of a BSC with crossover probability ϵ and capacity $1 - H_{\text{b}}(\epsilon)$. We conclude that the capacity of this channel does not increase by revealing the state to the receiver when the state is known causally at the transmitter.

Problem 4 — *Causal State Information Does Not Always Help*

If the state is not known at the transmitter and at the receiver, the equivalent channel law is

$$W(y|x) = \sum_{s} P_S(s) W(y|x,s).$$

For the given channel, it is easy to see that $W(y|x)$ is the law of a Z-channel with crossover probability ρ. Thus, the capacity is that of a Z-channel with crossover probability ρ.

We now show that this is also the capacity when the state is known causally at the transmitter. We know that the capacity in this case can be expressed as

$$\max_{P_T} I(T;Y),$$

where the maximum is over PMFs on the set of all mappings $t\colon \mathcal{S} \to \mathcal{X}$. For the given channel, the number of such mappings is 4. However, since the channel output is deterministically 0 when the state is "stuck", it does not matter what input symbol we use when the channel is "stuck". Thus, we only have to consider two mappings: $t_0(\text{“good”}) = 0$ and $t_1(\text{“good”}) = 1$. It can easily be verified that in this case, too, the channel from $T \in \{t_0, t_1\}$ to Y is a Z-channel with crossover probability ρ. We conclude that causal knowledge of the state does not increase the capacity of this channel.

Problem 5 — ***Receiver State Information Can Help***

When the state is known causally at the transmitter and at the receiver, the capacity is

$$\max_{P_{X|S}} I(X;Y|S).$$

For the channel from Problem 4, we have

$$\begin{aligned}
\max_{P_{X|S}} I(X;Y|S) &= \Pr[S = \text{“stuck”}] \cdot \max_{P_{X|S=\text{“stuck”}}} I(X;Y|S = \text{“stuck”}) \\
&\quad + \Pr[S = \text{“good”}] \cdot \max_{P_{X|S=\text{“good”}}} I(X;Y|S = \text{“good”}) \\
&= \Pr[S = \text{“stuck”}] \cdot 0 + \Pr[S = \text{“good”}] \cdot 1 \\
&= 1 - \rho,
\end{aligned}$$

which is in general larger than the capacity of a Z-channel with crossover probability ρ (the capacity when the state is known causally only at the transmitter). If for example $\rho = 0.5$, then $1 - \rho = 0.5$ bits per channel use, while the capacity of the Z-channel is approximately 0.322 bits per channel use.