Information Theory II

Spring Semester 2017

Prof. Dr. A. Lapidoth / Prof. Dr. S. Moser

Signal and Information Processing Laboratory
Institut für Signal- und Informationsverarbeitung

Exercise 11 of May 11, 2017

http://www.isi.ee.ethz.ch/teaching/courses/it2.html

Problem 1 — *Method of Types*

Let $\mathcal{X} = \{a, b, c\}$ be a ternary alphabet. Consider sequences $\mathbf{x} \in \mathcal{X}^5$ of length $n = 5$.

a) What is the exact number of types that $\mathbf{x}$ can take on, i.e., what is $|\mathcal{P}_5|$? Compare this number to the upper bounds derived in class.

b) What is the type of $\mathbf{x} = (b, c, c, a, b)$?

c) How many sequences have the same type as $\mathbf{x} = (b, c, c, a, b)$? Compare this number to the upper and the lower bounds derived in class. Are these bounds useful? Is there an easy way to improve the lower bound?

d) Assume that $\mathbf{X}$ is drawn IID Q where $Q(a) = 2/3$ and $Q(b) = Q(c) = 1/6$. What is the probability that $\mathbf{X} = (b, c, c, a, b)$? Compute it both directly and using types.

e) Still assuming that $\mathbf{X}$ is drawn IID Q, give an upper and a lower bound to the probability that $P_{\mathbf{X}} = P_{(b,c,c,a,b)}$. Can you compute this probability exactly?

Problem 2 — *Error Exponent for Universal Codes*

A universal code of rate R applied to an IID source of distribution Q achieves an error probability

$$P_{\mathrm{e}}^{(n)} \leq (n+1)^{|\mathcal{X}|} e^{-nD(P^* \| Q)},$$

where P^* achieves $\min D(P\|Q)$ over all P such that $H(P) \geq \mathsf{R}_n$.

a) For given Q and R_n, what form does P^* have?

Hint: You may use Lagrange multipliers to solve this part.

b) Let X be binary. Find the region of source distributions Q for which rate R is sufficient for the universal source code to achieve $P_{\mathrm{e}}^{(n)} \to 0$.

Problem 3 — *Large Deviations*

Let $X_1, X_2, \ldots, X_n$ be IID $\mathcal{N}(0, \sigma^2)$. Find the exponent in the probability that

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \geq \alpha^2.$$

Hint: Although the alphabet is infinite, you may use Sanov's theorem to solve this problem.