Information Theory II

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http://www.isi.ee.ethz.ch/teaching/courses/it2.html

Problem 1 — *Large Deviations*

Let $X_1, X_2, \ldots, X_n$ be IID random variables drawn according to the geometric distribution

$$\Pr[X = i] = (1-p)^{i-1}\,p, \quad i \in \{1, 2, \ldots\}.$$

For a given $\alpha > \frac{1}{p}$, find the exponent in the probability that $\frac{1}{n}\sum_{k=1}^{n} X_k \geq \alpha$.
Hint: Although the alphabet is infinite, you may use Sanov's theorem to solve this problem.

Problem 2 — *Sanov-Type Theorem for the Size of Type Classes*

Let $\mathcal{X}$ be a finite alphabet. Show that the number of sequences $\mathbf{x} \in \mathcal{X}^n$ satisfying $\frac{1}{n}\sum_{k=1}^{n} g(x_k) \geq \alpha$ is approximately equal to $e^{n\,\mathsf{H}^*}$, to first order in the exponent, for n sufficiently large, where

$$\mathsf{H}^* \triangleq \sup_{Q\in\mathcal{P}(\mathcal{X}):\ \sum_{x\in\mathcal{X}} Q(x)g(x)\geq\alpha} \mathsf{H}(Q).$$

More generally, prove the following Sanov-type theorem:

Theorem 1. *Let $\mathcal{F} \subseteq \mathcal{P}(\mathcal{X})$ be a set of probability distributions. Then*

$$\bigl|\mathcal{T}^n(\mathcal{F})\bigr| \leq (n+1)^{|\mathcal{X}|}\, e^{n \sup_{Q\in\mathcal{F}} \mathsf{H}(Q)}.$$

If in addition the set $\mathcal{F}$ is "nice", by which we mean that there exists a sequence of types in $\mathcal{F}$, $\{P_n \in \mathcal{F} \cap \mathcal{P}_n(\mathcal{X})\}$, such that

$$\liminf_{n\to\infty} \mathsf{H}(P_n) = \sup_{Q\in\mathcal{F}} \mathsf{H}(Q),$$

then

$$\liminf_{n\to\infty} \frac{1}{n}\log\bigl|\mathcal{T}^n(\mathcal{F})\bigr| \geq \sup_{Q\in\mathcal{F}} \mathsf{H}(Q),$$

i.e., we have

$$\boxed{\lim_{n\to\infty} \frac{1}{n}\log\bigl|\mathcal{T}^n(\mathcal{F})\bigr| = \sup_{Q\in\mathcal{F}} \mathsf{H}(Q).}$$

Problem 3 — *American Football*

Consider a very much simplified version of American football: assume that the score depends only on the total number of yards gained during all moves. The coach of each team has the choice of two basic strategies: passing or running. Associated with each strategy is a certain probability distribution on the number of yards gained: passing yields a large gain in yards, but only with small probability of success, while running leads to smaller gains, but with higher probability of success. Assume that the two strategies have the following distributions:

$$Q_{\text{pass}}(z) = \begin{cases} \frac{4}{5} & \text{for } z = 0, \\ \frac{1}{5} & \text{for } z = 10, \\ 0 & \text{otherwise} \end{cases}$$

for the strategy of passing, and

$$Q_{\text{run}}(z) = \begin{cases} \frac{1}{4} & \text{for } z \in \{1, 2, 3, 4\}, \\ 0 & \text{otherwise} \end{cases}$$

for the strategy of running.

Let $\{Z_k\}$ be an IID random process where Z_k denotes the number of yards gained in each move, $k = 1, \ldots, n$.

a) Compute the following relative entropy term:

$$\inf_{\tilde{Q} \in \mathcal{P}(\mathbb{N}_0)\colon\, \mathsf{E}_{\tilde{Q}}[Z] \geq 4} D(\tilde{Q} \| Q_{\text{pass}}),$$

where $\mathcal{P}(\mathbb{N}_0)$ stands for the set of all probability distributions with the nonnegative integers as alphabet.

b) Compute the following relative entropy term:

$$\inf_{\tilde{Q} \in \mathcal{P}(\mathbb{N}_0)\colon\, \mathsf{E}_{\tilde{Q}}[Z] \geq 4} D(\tilde{Q} \| Q_{\text{run}}).$$

Attention: Think first before you compute! You do not need Lagrange multipliers here!

c) Assume that shortly before the end of a game a team is behind by a considerable amount. It can only win if in the remaining n moves it achieves in total $4n$ yards. The coach decides to play on luck and only use the strategy of passing the ball. How big is the chance of still winning if n is large?

d) How big would the chance of winning be if the coach only used the strategy of running? Which strategy is better?

Problem 4 — ***Hypothesis Testing***

Let $X_1, X_2, \ldots, X_n$ be IID P under hypothesis H_0 and IID Q under hypothesis H_1 with

$$P(x) = \begin{cases} \frac{1}{2} & x = -1, \\ \frac{1}{4} & x = 0, \\ \frac{1}{4} & x = 1 \end{cases}$$

and

$$Q(x) = \begin{cases} \frac{1}{4} & x = -1, \\ \frac{1}{4} & x = 0, \\ \frac{1}{2} & x = 1. \end{cases}$$

Find the optimal error exponent for $\Pr(\text{Decide } H_1 | H_0 \text{ true})$ subject to $\Pr(\text{Decide } H_0 | H_1 \text{ true}) \leq \frac{1}{2}$.