Problem 1  
**Optimal Guessing**

Let $\rho > 0$, and let $X$ and $Y$ be distributed according to the joint PMF $P_{XY}$:

\[
\begin{array}{c|cc}
   & y = 0 & y = 1 \\
 x = 0 & 0.1 & 0.2 \\
x = 1 & 0.3 & 0.4 \\
\end{array}
\]

a) Compute $E[G^*(X)^\rho]$.

b) Compute $E[G^*(X|Y)^\rho]$. What do you observe?

c) Let $X_1, X_2, \ldots, X_n$ be IID $P_X$. Compute $\lim_{n \to \infty} \frac{1}{n} \log E[G^*(X^n)^\rho]$.

Problem 2  
**Rényi Entropy**

The Rényi entropy of order $\alpha$ of a discrete chance variable $X$ taking values in $\mathcal{X}$ according to the PMF $P_X$ is defined for $\alpha > 0$ and $\alpha \neq 1$ as

\[
H_\alpha(X) \triangleq \frac{1}{1-\alpha} \log \sum_{x \in \mathcal{X}} P_X(x)^\alpha.
\]

a) Show that $H_\alpha(X) \geq 0$ with equality if and only if $X$ is deterministic.

*Hint: Treat the cases $\alpha \in (0, 1)$ and $\alpha > 1$ separately.*

b) Show that the Rényi entropy approaches Shannon entropy as $\alpha$ approaches unity, i.e.,

\[
\lim_{\alpha \to 1} H_\alpha(X) = H(X).
\]

Problem 3  
**Rényi Divergence**

The Rényi divergence of order $\alpha$ between two PMFs $P$ and $Q$ is defined for $\alpha > 0$ and $\alpha \neq 1$ as

\[
D_\alpha(P\|Q) \triangleq \frac{1}{\alpha - 1} \log \sum_{x \in \mathcal{X}} P(x)^\alpha Q(x)^{1-\alpha}
\]

with the convention that for $\alpha > 1$, we read $P(x)^\alpha Q(x)^{1-\alpha}$ as $\frac{P(x)^\alpha}{Q(x)^{\alpha-1}}$ and say that $\frac{0}{0} = 0$ and $\frac{p}{\infty} = \infty$ for $p > 0$.

a) Show that the Rényi divergence approaches relative entropy as $\alpha$ approaches unity, i.e.,

\[
\lim_{\alpha \to 1} D_\alpha(P\|Q) = D(P\|Q).
\]

b) Let $U$ be the uniform distribution over $\mathcal{X}$. What is $D_\alpha(P\|U)$?
Problem 4  

**Guessing with Chosen Side Information**

Let $X_1, X_2, \ldots, X_n$ be IID $P_X$, and let $R \geq 0$. Given the side information $Y_n = f_n(X^n)$, where $f_n$ is a function from $X^n$ to $\{1, \ldots, 2^{nR}\}$, the decoder has to guess the sequence $X^n$.

a) Show that if $R < H_{\frac{1}{1+\rho}}(X)$, then for every sequence of functions $(f_1, f_2, \ldots)$,

$$
\lim_{n \to \infty} E[G^*(X^n|f_n(X^n))^\rho] = \infty.
$$

b) Show that if $R > H_{\frac{1}{1+\rho}}(X)$, then there exists a sequence of functions $(f_1, f_2, \ldots)$ for which

$$
\lim_{n \to \infty} E[G^*(X^n|f_n(X^n))^\rho] = 1.
$$

*Hint: Partition every type class into $\lceil \frac{2^{nR}}{2(n+1)^{|X|}} \rceil$ subsets. For $n$ large enough, the total number of these sets does not exceed $2^{nR}$. To bound the number of guesses, you can use the inequality $[\xi]^\rho \leq 1 + 2^\rho \xi^\rho$, which holds for all $\rho > 0$ and $\xi \geq 0$.**

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