Problem 1

Strong Law of Large Numbers

a) i) $Z_k$ is plotted in Figure 1.

![Plot of $Z_k$](image)

Figure 1: Plot of $Z_k$.

ii) The limit does not exist, since the sequence is oscillating between (almost) $-1$ and (almost) $1$ for very large $k$.

iii) The limsup is defined as the limit of a new sequence $\tilde{Z}_n$ where

$$\tilde{Z}_n = \sup_{k \geq n} Z_k,$$

Note that by definition $\tilde{Z}_n$ is a nonincreasing sequence, hence either tends to a limit or goes to $-\infty$. In our case $\tilde{Z}_n = 1$ for all $n$, and therefore

$$\lim_{k \to \infty} Z_k = \lim_{n \to \infty} \sup_{k \geq n} Z_k = 1.$$

iv) Similarly, the liminf is defined as the limit of the new sequence

$$\tilde{Z}_n = \inf_{k \geq n} Z_k.$$

In our case $\tilde{Z}_n = -1$ for all $n$, and hence $\lim_{k \to \infty} Z_k = -1$. 
b) Since $a_n \to a$, for every $\epsilon > 0$ there exists a number $n_\epsilon$ such that $|a_n - a| \leq \epsilon$ for all $n \geq n_\epsilon$. Hence, for $n \geq n_\epsilon$

\[ |b_n - a| = \left| \frac{1}{n} \sum_{k=1}^{n} a_k - a \right| \]
\[ = \left| \frac{1}{n} \sum_{k=1}^{n} a_k - \frac{1}{n} \sum_{k=1}^{n} a \right| \]
\[ = \left| \frac{1}{n} \sum_{k=1}^{n} (a_k - a) \right| \]
\[ \leq \frac{1}{n} \sum_{k=1}^{n} |a_k - a| \quad \text{(triangle inequality)} \]
\[ = \frac{1}{n} \sum_{k=1}^{n} |a_k - a| + \frac{1}{n} \sum_{k=n_\epsilon}^{n} |a_k - a| \]
\[ \leq \frac{1}{n} \sum_{k=1}^{n-1} |a_k - a| + \frac{1}{n} \sum_{k=n_\epsilon}^{n} \epsilon \quad \text{(by definition of } n_\epsilon) \]
\[ = \frac{1}{n} \sum_{k=1}^{n-1} |a_k - a| + \frac{n - n_\epsilon + 1}{n} \epsilon \]
\[ \leq \frac{1}{n} \sum_{k=1}^{n-1} |a_k - a| + \epsilon. \]

Since the first term tends to 0 as $n \to \infty$, we can make $n$ large enough such that $|b_n - a| \leq 2\epsilon$. Hence $b_n \to a$ as $n \to \infty$.

c)

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n+\ell} X_k = \lim_{n \to \infty} \frac{1}{n} \left( \sum_{k=1}^{n+\ell} X_k - \sum_{k=1}^{\ell} X_k \right) \]
\[ = \lim_{n \to \infty} \frac{n + \ell}{n} \left( \frac{1}{n + \ell} \sum_{k=1}^{n+\ell} X_k - \frac{1}{n + \ell} \sum_{k=1}^{\ell} X_k \right) \]
\[ = X^*. \]

d) i) We write

\[ X_k = \max\{X_k, 0\} - \max\{-X_k, 0\} \]

and note that

\[ \left( X_k \in \mathcal{L}_1 \right) \iff \left( \max\{X_k, 0\} \in \mathcal{L}_1, \max\{-X_k, 0\} \in \mathcal{L}_1 \right) \]

and thus

\[ \mathbb{E}[X_k] = \mathbb{E}[\max\{X_k, 0\}] - \mathbb{E}[\max\{-X_k, 0\}]. \]

So, we see that if the SLLN holds for nonnegative RVs (i.e., it holds for $\max\{X_k, 0\}$ and $\max\{-X_k, 0\}$), then it also holds for $X_k$. 

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ii) Since by definition $X_\ast \leq X^\ast$, it follows from $X_\ast \geq X^\ast$ almost surely that $X_\ast = X^\ast$ almost surely.

iii) We assume that $X_k \geq 0$. Hence, if $E[X^\ast] = 0$, then $X_k = 0$ almost surely for all but a finite number of $k$ and the claim is true trivially.

iv) Since $X_\ast \leq X^\ast$, we have $E[X_\ast] \geq E[X^\ast]$ if, and only if, $X_\ast = X^\ast$ almost surely.

e) In d) we have argued that if $E[X_\ast] \geq E[X^\ast]$, then $X_\ast = X^\ast$ almost surely, establishing convergence. From c) we know that $X_\ast$ and $X^\ast$ both are constants almost surely. Hence, if $E[X_\ast] \geq \mu \geq E[X^\ast]$, then this also means that $c = \mu$, i.e., we have established the SLLN.

f) i) We define $Y_k \triangleq m - \min\{X_k, m\}$. Then

$$Y^\ast = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} Y_k$$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (m - \min\{X_k, m\})$$

$$= \lim_{n \to \infty} \left( m - \frac{1}{n} \sum_{k=1}^{n} \min\{X_k, m\} \right)$$

$$= m - \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \min\{X_k, m\}$$

$$= m - \left( \min\{X_k, m\} \right)_{\ast}.$$ 

ii) The first inequality can be shown as follows:

$$E[X_\ast] = E \left[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} X_k \right]$$

$$\geq E \left[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \min\{X_k, m\} \right]$$

$$= E \left[ \left( \min\{X_k, m\} \right)_{\ast} \right].$$

To show the second inequality we use our assumption that $E[Y_1] \geq E[Y^\ast]$:

$$m - E[\min\{X_1, m\}] = E[Y_1] \geq E[Y^\ast] = m - E \left[ \left( \min\{X, m\} \right)_{\ast} \right]$$

and therefore

$$E \left[ \left( \min\{X, m\} \right)_{\ast} \right] \geq E[\min\{X_1, m\}].$$

iii) Since $\min\{X_1, m\} \leq X_1$ and $X_1 \in L_1$, it follows by the dominated convergence theorem that $E[\min\{X_1, m\}] \to E[X_1] = \mu$ as $m \to \infty$. So, using f-ii) (that relies on the assumption that for every nonnegative sequence $\{X_k\}$ it holds that $E[X_1] \geq E[X^\ast]$), we have

$$E[X_\ast] \geq E[\min\{X_1, m\}] \to \mu.$$

Hence, it only remains to show that for any nonnegative sequence $\{X_k\}$ it is always true that $E[X_1] \geq E[X^\ast]$.
g) Note that by the IID-property of \( \{X_k\} \) we have
\[
\mu = \mathbb{E}[X_1] = \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}[X_k] = \mathbb{E}\left[ \frac{1}{n} \sum_{k=1}^{n} X_k \right],
\]
which holds for any choice of \( n \). Hence,
\[
\mu = \lim_{n \to \infty} \mathbb{E}\left[ \frac{1}{n} \sum_{k=1}^{n} X_k \right].
\]

Note further that if we can show that \( \mu \geq \alpha \) where \( \alpha \) is any constant, \( \alpha \in (0, \mathbb{E}[X^*]) \), then \( \mu \geq \mathbb{E}[X^*] \) since the choice of \( \alpha \) is arbitrary.

h) Fix some \( \alpha < \mathbb{E}[X^*] \) and assume by contradiction that \( \alpha \leq \frac{1}{n} \sum_{k=1}^{n} X_k \) for only finitely many \( n \). Then there exists an \( n_0 \) such that
\[
\alpha > \frac{1}{n} \sum_{k=1}^{n} X_k \quad \forall n \geq n_0,
\]
i.e.,
\[
\alpha \geq \sup_{n \geq n_0} \frac{1}{n} \sum_{k=1}^{n} X_k \quad \text{(upper bound holds for all } n \geq n_0, \text{ so it also holds for the supremum)}
\]
\[
\geq \limsup_{t \to \infty} \sup_{n \geq t} \frac{1}{n} \sum_{k=1}^{n} X_k \quad \text{(reducing the range of the supremum)}
\]
\[
= \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} X_k \quad \text{(definition of } \limsup \text{)}
\]
\[
= X^* \quad \text{(definition of } X^* \text{)}
\]
and hence
\[
\alpha \geq \mathbb{E}[X^*],
\]
which is a contradiction to \( \alpha < \mathbb{E}[X^*] \). Hence, \( \alpha \leq \frac{1}{n} \sum_{k=1}^{n} X_k \) must hold for infinitely many \( n \).

i) It is easiest to firstly study a simple example: Let
\[
(X_1, X_2, \ldots, X_n) = (5, 1, 3, 7, 2, 4, 3, 1),
\]
i.e., we have \( n = 8 \). Let’s choose \( \alpha = 3.1 \). Now we get
- \( A_1 = \{1\} \) because \( X_1 = 5 \geq 3.1 \);
- \( A_2 = \{2, 3, 4\} \) because \( \frac{1}{3}(X_2 + X_3 + X_4) = \frac{1+2+3}{3} = \frac{6}{3} \geq 3.1 \);
- \( 5 \notin A_3 \) because \( 2 < 3.1, \frac{2+4}{2} < 3.1, \frac{2+4+3}{3} < 3.1, \frac{2+4+3+1}{4} < 3.1 \);
- \( A_3 = \{6\} \) because \( 4 \geq 3.1 \);
- \( 7 \notin A_4 \) because \( 3 < 3.1, \frac{3+1}{2} < 3.1 \);
- \( 8 \notin A_4 \) because 1 < 3.1;
- hence \( J = 3 \).
Hence, we see that the indices not in $A_1 \cup A_2 \cup A_3$ are $k \in \{5, 7, 8\}$. They are not member of any $A_j$ because $L(5) > 4$, $L(7) > 2$, and $L(8) > 1$.

j) So we note that by construction any $k \notin A_1 \cup \cdots \cup A_J$ must satisfy that the length of a nice set starting at $k$ must be longer than the remaining available indices, i.e., $k > n - k + 1$. If this were not the case, then we could have generated a nice set at $k$ and hence we would have done it.

k) The first inequality follows from Part j) because for every $k \notin A_1 \cup \cdots \cup A_J$ we know that $L(k) > n - k + 1$ is satisfied, making sure that the sum is at least equal to $N$; the subsequent equality follows by linearity; and the final equality follows by definition of the expected value.

l) The first inequality follows because some of the indices in $\{1, \ldots, n\}$ are dropped in the summation and because $X_k \geq 0$; the subsequent equality from the fact that the nice intervals $A_j$ are disjoint; the subsequent inequality from the fact that the average sum of $X_k$ over each nice interval is lower bounded by $\alpha$ (by construction!); the subsequent equality follows from arithmetic changes; and the last equality from the definition of $N$.

m) This follows from Part h).

n) The first inequality follows from Part l); the subsequent equality from linearity; the subsequent inequality from Part k); the subsequent equality from stationarity (note that we have assumed the sequence to be IID which is a stronger assumption than stationarity!); then we relabel the summation variable; and the last step follows from the Cesáro Mean Lemma using Part m).