Optimum Frame Synchronization

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***Abstract*—This paper considers the optimum method for locating a sync word periodically imbedded in binary data and received over the additive white Gaussian noise channel. It is shown that the optimum rule is to select the location that maximizes the sum of the correlation and a correction term. Simulations are reported that show approximately a 3-dB improvement at interesting signal-to-noise ratios compared to a pure correlation rule. Extensions are given to the "phase-shift keyed (PSK) sync" case where the detector output has a binary ambiguity and to the case of Gaussian data.**

I. Introduction

THE MOST widely used method for providing frame synchronization in a binary signaling scheme is to insert a fixed binary pattern or "sync word" periodically into the data stream. On the assumption that symbol synchronization has already been obtained, the receiver obtains frame synchronization by locating the position of the sync word in the received data stream.

In his pioneering work [1] on frame synchronization, Barker assumed that the sync word would be located by passing the received digits through a "pattern recognizer," which was simply a device to correlate successive L-digit segments of the received sequence with the L-digit sync word. The segment giving the maximum correlation would be taken as the location of the sync word. Virtually all subsequent work on frame synchronization has assumed this same correlation decision rule, perhaps for simplicity and perhaps in the belief that this decision rule was optimal. In his encyclopedic coverage of synchronization, Stiffer [2, pp. 499–502] recognizes that the data surrounding the sync word should be taken into account by an optimal decision rule, but indicates that the analysis becomes intractable and that the resulting true optimal decision rule would be impractical to implement.

In this paper, we derive the optimal decision rule for locating the sync word on the additive white Gaussian noise channel and show that the effect of the data is merely to add a "correction" term to the correlator output so that the optimum rule is nearly as simple to implement as the ordinary correlation rule. This derivation is given in Section II for the standard case where the receiver can make tentative bit decisions. Section III gives the necessary modification for the "phase-shift keyed (PSK) sync" case where the bit values are ambiguous until after frame synchronization is obtained. Section IV contains the results of simulations comparing the performance of the optimum rule and the correlation rule. Section V gives a derivation of the optimum sync word locating rule when the data, rather than being random binary digits, are Gaussian random variables as might be the case in some pulse-amplitude modulation schemes.

It should be emphasized that our analysis applies only to the case of a sync word periodically imbedded into a data stream, which is the usual case in space telemetry. Specifically, it does not apply to the "one-shot" synchronization problem where the sync word is prefixed to the data stream and is itself preceded either by no signal or by a periodic 1–0 pattern. It remains as an interesting open problem to find the optimum synchronization rule for this one-shot case.

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II. Derivation of the Optimal Sync-Word Locating Rule

Let N denote the frame length, i.e., each L-digit sync word is followed by N–L random binary-data bits. We assume that the receiver is to process an N digit span of the received sequence in order to locate the sync word contained therein. If n such spans are actually to be used, the problem reduces to the above for a frame length of nN digits and a sync word of length nL.

Let $\boldsymbol{r} = (r_0, r_1, \cdots, r_{N-1})$ denote the received span to be processed where each r_i is the detector output over one of the assumed-known bit intervals. The sync word is *a priori* equally likely to begin in any of the N positions of $\boldsymbol{r}$. We will consider digit r_0 to follow digit r_{N-1} so as to account for the case when the sync word begins somewhere in the last L-1 digits of $\boldsymbol{r}$ and all subscripts on received digits will hereafter be taken modulo N. For example, r_{N+2} is the digit r_2.

Let $\boldsymbol{s} = (s_0, s_1, \cdots, s_{L-1})$, where each s_i is either $+1$ or -1, be the sync word and let $\boldsymbol{d} = (d_L, d_{L+1}, \cdots, d_{N-1})$ denote N–L random data bits where the d_i are statistically independent random variables satisfying $\Pr[d_i = +1] = \Pr[d_i = -1] = \frac{1}{2}$. Consider next the concatenation $\boldsymbol{sd} = (s_0, s_1, \cdots, s_{L-1}, d_L, \cdots, d_{N-1})$. Let T be the cyclic shift operator defined by $T(\boldsymbol{sd}) = (d_{N-1}, s_0, \cdots, s_{L-1}, d_L, \cdots, d_{N-2})$. If the sync word actually begins in digit r_m of $\boldsymbol{r}$, we can express the received segment as

$$\boldsymbol{r} = \sqrt{E}\, T^m(\boldsymbol{sd}) + \boldsymbol{n} \tag{1}$$

where each received digit would have value either $+\sqrt{E}$

or $-\sqrt{E}$ in the absence of noise, and where $\boldsymbol{n} = (n_0, n_1, \cdots, n_{N-1})$ is the contribution of the additive white Gaussian noise to the detector outputs. The components n_i are statistically independent Gaussian random variables with 0 mean and variance $N_0/2$ where N_0 is the one-sided noise spectral density. Let $\boldsymbol{\rho} = (\rho_0, \rho_1, \cdots, \rho_{N-1})$ denote the actual value assumed by the random vector $\boldsymbol{r}$. Then the optimum [in the sense of maximizing the probability of correctly locating the sync word] decision rule is to choose the estimate of m as the value μ, $0 \leq \mu < N$, which maximizes $S_1 = \Pr[m = \mu \mid \boldsymbol{r} = \boldsymbol{\rho}]$, which by the mixed Bayes' rule [3, p. 75] for events and random variables becomes $S_1 = p_r(\boldsymbol{\rho} \mid m = \mu) \Pr[m = \mu]/p_r(\boldsymbol{\rho})$. Here and hereafter, lower case p denotes the density function of the subscripted random variable. Since $\Pr[m = \mu] = 1/N$ for all μ, we may equivalently maximize $S_2 = p_r(\boldsymbol{\rho} \mid m = \mu)$. Letting $\boldsymbol{\delta} = (\delta_L, \delta_{L+1}, \cdots, \delta_{N-1})$, where each δ_i is either $+1$ or -1, denote a possible value of the random data vector $\boldsymbol{d}$, we have

$$S_2 = \sum_{\text{all } \boldsymbol{\delta}} p_r(\boldsymbol{\rho} \mid \boldsymbol{d} = \boldsymbol{\delta}, m = \mu) \Pr(\boldsymbol{d} = \boldsymbol{\delta}). \tag{2}$$

Since $\Pr(\boldsymbol{d} = \boldsymbol{\delta}) = 2^{-(N-L)}$ for all $\boldsymbol{\delta}$, we may equivalently maximize

$$S_3 = \sum_{\text{all } \boldsymbol{\delta}} p_r(\boldsymbol{\rho} \mid \boldsymbol{d} = \boldsymbol{\delta}, m = \mu),$$

which upon making use of (1) becomes

$$S_3 = \sum_{\text{all } \boldsymbol{\delta}} p_n(\boldsymbol{\rho} - \sqrt{E}\, T^{\mu}(\boldsymbol{s\delta})). \tag{3}$$

By the Gaussian assumption on $\boldsymbol{n}$, we have

$$p_n(\boldsymbol{\rho} - \sqrt{E}\, T^{\mu}(\boldsymbol{s\delta})) = (2\pi)^{-N/2} \cdot \left[\prod_{i=0}^{L-1} e^{-(\rho_{i+\mu} - \sqrt{E} s_i)^2/N_0}\right] \prod_{i=L}^{N-1} e^{-(\rho_{i+\mu} - \sqrt{E}\delta_i)^2/N_0}.$$

Substituting this expression into (3) and removing all factors independent of μ, we may equivalently maximize

$$S_4 = \sum_{\text{all } \boldsymbol{\delta}} \left[\prod_{i=0}^{L-1} e^{2\sqrt{E}\rho_{i+\mu}s_i/N_0}\right] \prod_{i=L}^{N-1} e^{2\sqrt{E}\rho_{i+\mu}\delta_i/N_0}. \tag{4}$$

Carrying out the summation in (4) noting that each δ_i takes on only the values $+1$ and -1, we obtain

$$S_4 = \left[\prod_{i=0}^{L-1} e^{2\sqrt{E}\rho_{i+\mu}s_i/N_0}\right] \prod_{i=L}^{N-1} 2 \cosh (2\sqrt{E}\, \rho_{i+\mu}/N_0).$$

Taking logarithms, we can equivalently maximize

$$S_5 = \sum_{i=0}^{L-1} 2\sqrt{E}\, \rho_{i+\mu}s_i/N_0 + \sum_{i=L}^{N-1} \log_e \cosh (2\sqrt{E}\, \rho_{i+\mu}/N_0).$$

Noting that

$$\sum_{i=0}^{N-1} \log_e \cosh (2\sqrt{E}\, \rho_{i+\mu}/N_0)$$

is a sum over all components of $\boldsymbol{\rho}$ and hence is independent of μ, we may subtract this sum from S_5 without affecting the maximization to give

$$S_6 = \sum_{i=0}^{L-1} 2\sqrt{E}\, \rho_{i+\mu}s_i/N_0 - \sum_{i=0}^{L-1} \log_e \cosh (2\sqrt{E}\, \rho_{i+\mu}/N_0)$$

as the quantity to be maximized by choice of μ in the optimum decision rule. Slightly rewriting, we summarize as follows.

Optimum Rule for Locating the Sync Word: Given the received segment $\boldsymbol{\rho}$, take the estimate of the sync word location m to be the value of μ, $0 \leq \mu < N$, which maximizes the statistic

$$S = \sum_{i=0}^{L-1} s_i\rho_{i+\mu} - \sum_{i=0}^{L-1} f(\rho_{i+\mu}) \tag{5}$$

where

$$f(x) = (N_0/2\sqrt{E}) \log_e \cosh (2\sqrt{E}\, x/N_0). \tag{6}$$

It should be noted that the first summation in (5) is the ordinary correlation. The second summation represents a kind of voltage or energy correction required to account for the random data surrounding the sync word. It should also be clear that the optimum statistic S is nearly as easy to calculate as the correlation above, particularly in the practical case where the detector outputs are quantized to 8 or 16 values (3- or 4-bit quantization.) In this case, only a small number of values of the function f in (6) need be stored for use in forming the correction term.

Additional insight into the nature of the optimum statistic S can be gained by examining its form in the limiting cases of very high and very low signal-to-noise ratios.

When $E/N_0 \gg 1$, the argument of the cosh in (6) is much greater than 1 with high probability so that we may approximate cosh (y) as $(\frac{1}{2})e^{|y|}$. Using this approximation in (6), we obtain

$$S = \sum_{i=0}^{L-1} s_i\rho_{i+\mu} - \sum_{i=0}^{L-1} |\rho_{i+\mu}|. \tag{7}$$

Note that whenever si and $\rho_{i+\mu}$ agree in sign, their contribution to the first summation in (7) is exactly cancelled by the term $-|\rho_{i+\mu}|$ in the second summation. Thus, only negatively correlated terms contribute to the statistic S and the optimum decision rule reduces to choosing that location μ for the sync word that yields the least total negative correlation.

When $E/N_0 \ll 1$, the argument of the cosh in (6) is much smaller than 1 with high probability so that we may closely approximate $\log_e \cosh(y)$ by the first term in its Maclaurin series expansion, namely $(\frac{1}{2})y^2$. Using this approximation in (6), we obtain

$$S = \sum_{i=0}^{L-1} s_i \rho_{i+\mu} - \frac{\sqrt{E}}{N_0} \sum_{i=0}^{L-1} \rho_{i+\mu}^2 \tag{8}$$

as the statistic to be maximized by the optimum decision rule. From the form of the second summation in (8), we see that the correction term is an energy correction in this small signal-to-noise ratio case.

III. PSK Frame Synchronization

As Stiffler has noted [2, p. 372], when a binary PSK signal is demodulated using a carrier reference derived from the modulated signal, there is a binary ambiguity in the detector output. With probability $\frac{1}{2}$, the detector output will be $r_i = t_i + n_i$ where t_i is the transmitted signal ($\sqrt{E}\, s_i$ or $\sqrt{E}\, d_i$), and with probability $\frac{1}{2}$ the detector output will be $r_i = -t_i - n_i$. When this ambiguity is included in the analysis, a derivation very similar to that in Section II, the details of which will be omitted here, leads to the following.

Optimum Rule for PSK Frame Sync: Given the received segment $\boldsymbol{\rho}$, take the estimate of the sync-word location m to be the value of μ, $0 \leq \mu < N$, which maximizes the statistic

$$S = \log_e \cosh (P) - \sum_{i=0}^{L-1} \log_e \cosh (2\sqrt{E}\, \rho_{i+\mu}/N_0) \tag{9}$$

where

$$P = (2\sqrt{E}/N_0) \sum_{i=0}^{L-1} s_i \rho_{i+\mu}. \tag{10}$$

A simple approximate form for this statistic is readily obtained. The correlation sum P can be expected to be quite large in general so that the approximation $\log_e \cosh(P) = |P| - \log_e 2$ will be quite accurate. Using this approximation, we obtain as the statistic to be maximized

$$S' = \left| \sum_{i=0}^{L-1} s_i \rho_{i+\mu} \right| - \sum_{i=0}^{L-1} f(\rho_{i+\mu}) \tag{11}$$

where $f(\ )$ is given in (6). We note that the usual correlation rule for PSK frame synchronization is just to choose μ to maximize the first summation in (11). Again we see the optimal decision rule adds a correction term to this usual statistic.

IV. Simulation Results

The basic question remaining is whether the optimum decision rule for frame synchronization provides significantly better performance than the ordinarily used but suboptimum correlation method. A theoretical comparison of performances is ruled out by the complicated relationship between probability of incorrect synchronization and channel signal-to-noise ratio. For this reason, a Monte Carlo simulation was performed to obtain sufficient empirical data for a performance comparison for channel signal-to-noise ratios E/N_0 near unity which is the range of practical interest in space telemetry.

To conform to usual communications practice, the detector outputs r_i in the simulation were quantized to 16 levels, and it was verified that 8 levels gave essentially the same results. The quantized values used were

$$(2j - 1)\sqrt{E}/6, \qquad -8 < j \leq 8, \tag{12}$$

and the quantization boundaries were taken halfway between adjacent quantization values. Simulations were performed using several different sync words and frame lengths as well as signal-to-noise ratios. Table I shows the results of standard frame synchronization by both the optimum and the correlation rules for two Barker sequences and one Neuman–Hofman [4] sequence as sync words and for three different signal-to-noise ratios. The same noise and data sequences were used with all sync sequences. Inspection of this table reveals a 3-dB advantage for the optimum decision rule for E/N_0 about 1 as seen by the fact that the optimum decision rule performs the same at $E/N_0 = 1$ as does the correlation rule at $E/N_0 = 2$. (This quite precise gain of 3 dB was obtained in all the simulations performed that cover a wide range of sync-word lengths and frame lengths.) The gain becomes less as the channel becomes more noisy.

It should be noted from Table I that the Neuman–Hofman sequence outperforms the Barker sequence of the same length as a sync word. The Neuman–Hofman sequence was designed so as to maximize performance with a correlation decision rule. This simulation also shows that it is a good choice for a sync word to be used with the optimum decision rule.

Table II shows the results of PSK frame synchronization for the same sequences and signal-to-noise ratios as in Table I. The general nature of the results are the same with the optimum decision rule at $E/N_0 = 1$ showing the same 3-dB improvement over the correlation rule. It should be mentioned that the "optimum" rule used in the simulations reported in Table II was actually the rule using the approximately optimum statistic S' of (12). This decision rule is indiscernible from the true optimum decision rule.

V. Gaussian Data

For the sake of completeness, we now consider the case when the data digits d_i, rather than being limited to values of $+1$ and -1, are instead statistically independent zero-mean Gaussian random variables with variance unity. Such a situation might describe digitized voice transmission with pulse-amplitude modulation (PAM). To derive the optimum sync-word location rule, we note that the analysis in Section II up to (2) is unchanged

TABLE I

PERCENTAGE OF ERRONEOUSLY SYNCHRONIZED FRAMES IN 100 SYNCHRONIZATION TRIALS USING THE OPTIMUM (OPT) AND THE CORRELATION (COR) RULES FOR LOCATING THE SYNC WORD

Sync Word	$E/N_0 = \frac{1}{2}$ OPT	$E/N_0 = \frac{1}{2}$ COR	$E/N_0 = 1$ OPT	$E/N_0 = 1$ COR	$E/N_0 = 2$ OPT	$E/N_0 = 2$ COR
$L = 13$, $N = 91$ Barker sequence $s = (1, 1, 1, 1, 1, -1, -1, 1, 1, -1, 1, -1, 1,)$	0.31	0.42	0.09	0.19	0.00	0.08
$L = 13$, $N = 91$ Neuman–Hofman sequence $s = (-1, -1, -1, -1, -1, -1, 1, 1, -1, -1, 1, -1, 1)$	0.28	0.32	0.07	0.18	0.00	0.07
$L = 7$, $N = 28$ Barker sequence $s = (1, -1, 1, 1, -1, -1, -1,)$	0.40	0.45	0.21	0.32	0.09	0.22

TABLE II

PERCENTAGE OF ERRONEOUSLY SYNCHRONIZED FRAMES IN 100 TRIALS FOR PSK FRAME SYNCHRONIZATION USING THE OPT AND THE COR RULES FOR LOCATING THE SYNC WORD

Sync Word	$E/N_0 = \frac{1}{2}$ OPT	$E/N_0 = \frac{1}{2}$ COR	$E/N_0 = 1$ OPT	$E/N_0 = 1$ COR	$E/N_0 = 2$ OPT	$E/N_0 = 2$ COR
$L = 13$, $N = 91$ Barker sequence $s = (1, 1, 1, 1, 1, -1, -1, 1, 1, -1, 1, -1, 1)$	0.39	0.47	0.14	0.27	0.00	0.12
$L = 13$, $N = 91$ Neuman–Hofman sequence $s = (-1, -1, -1, -1, -1, -1, 1, 1, -1, -1, 1, -1, 1)$	0.39	0.49	0.14	0.24	0.00	0.13
$L = 7$, $N = 28$ Barker sequence $s = 1, -1, 1, 1, -1, -1, -1)$	0.63	0.63	0.37	0.46	0.21	0.40

and (2) becomes replaced by

$$S_2 = \int_{-\infty}^{+\infty} p_r(\boldsymbol{\rho} \mid \mathbf{d} = \boldsymbol{\delta}, m = \mu) p_d(\boldsymbol{\delta})\, d\boldsymbol{\delta}$$

$$= \int_{-\infty}^{+\infty} p_n(\boldsymbol{\rho} - \sqrt{E}\, T^{\mu}(\boldsymbol{s\delta})) p_d(\boldsymbol{\delta})\, d\boldsymbol{\delta}. \qquad (13)$$

Using the assumption that the components of $\boldsymbol{n}$ are statistically independent Gaussian random variables with mean 0 and variance $N_0/2$ and that the components of $\boldsymbol{d}$ are statistically independent Gaussian random variables with mean 0 and variance 1, we obtain

$$S_2 = (2\pi)^{-N/2}\left[\prod_{i=0}^{L-1} e^{-(\rho_{i+\mu}-\sqrt{E}s_i)^2/N_0}\right] \cdot \prod_{j=L}^{N-1} \int_{-\infty}^{+\infty} e^{-(\rho_{j+\mu}-\sqrt{E}\delta_j)^2/N_0 - \delta_j^2/2}\, d\delta_j.$$

Carrying out the indicated integrations and dropping factors independent of μ, we may equivalently maximize

$$S_3 = -\sum_{i=0}^{L-1} \rho_{i+\mu}^2/N_0 + (2\sqrt{E}/N_0) \cdot \sum_{i=0}^{L-1} s_i\rho_{i+\mu} - \frac{1}{N_0 + 2E}\sum_{i=L}^{N-1} \rho_{i+\mu}^2.$$

Adding the quantity

$$\frac{1}{N_0 + 2E}\sum_{i=0}^{N-1} \rho_{i+\mu}^2,$$

which is independent of μ, we may equivalently maximize the statistic

$$S = \sum_{i=0}^{L-1} s_i\rho_{i+\mu} - \frac{\sqrt{E}}{N_0 + 2E}\sum_{i=0}^{L-1} \rho_{i+\mu}^2. \qquad (14)$$

Once again we see that the statistic to be maximized by the optimum sync-word location rule consists of the ordinary correlation together with a correction term. For Gaussian data, the correction term is again seen to be a true energy correction.

VI. SUMMARY

This paper has considered the problem of obtaining an optimal estimate of the location of the sync word in a data frame for binary data in the standard case and in the PSK sync case and also for Gaussian data samples. In all cases it was shown that the optimum decision rule is to maximize a statistic that is the sum of two terms, the first being the usual correlation and the second being an energy correction that takes into account the fact that the sync word is imbedded in data. It was verified by simulation that the optimum rule provides about a 3-dB advantage over the ordinary correlation rule in the interesting case of signal-to-noise ratios E/N_0 near unity. It was also noted that the optimum decision rule is only slightly more complicated to implement than the correlation rule.

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