CODING AND MODULATION IN DIGITAL COMMUNICATIONS

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ABSTRACT

The use of coding in digital communications, to be effective, requires that the modulation system be designed on an unconventional basis. Rather than using "error probability" as the modulation criterion, this paper argues that the appropriate modulation criterion is the "cut-off rate" R_o of the discrete channel which the modulation system creates. It is then shown that the R_o criterion leads to a rich "communication theory" of its own in which the optimality of the simplex signal set can be proved (rather than conjectured) and in which "soft decision" demodulators can be systematically designed. At the same time, the R_o criterion leads to a modulation system compatible with the use of effective coding techniques in an overall efficient digital communication system.

I. Introduction

Information theory has been around for a sufficiently long time that its model of a digital communication system, given in Fig. 1, no longer startles the communications engineer. Most communications engineers would now agree that encoding and modulation are both aspects of the signal design problem, and that demodulation and decoding are likewise both aspects of the signal detection problem. Yet coding techniques, in spite of the great research effort and resultant large literature thereon, are seldom used in practical systems. Ten years ago, the "experts" held that coding was an interesting intellectual game that lacked practical relevance; their motto was then "Coding is dead." In the meantime, coding has been used with spectacular success in deep-space communications. Now the "experts" are saying that the deep-space channel, i.e. the additive white Gaussian noise channel constrained in energy but not in bandwidth, is the only channel where coding techniques will ever be practical. The motto is now "Coding is dead except for the deep-space channel."

In this paper, we shall argue that this pessimism about coding stems from the failure of communications engineers truly to accept the basic model of Fig. 1.

In Section III, we define certain concepts implicit in Fig. 1. We then argue in Section IV that acceptance of the Fig. 1 model forces the communications engineer to discard "error probability" as a modulation design criterion in favor of what we call the "R_o criterion" where R_o is the cut-off rate of the discrete memoryless channel created by the modulation system. Although the R_o criterion is intended to lead to design of modulation systems which are compatible with effective coding systems, we show in the remainder of the paper that the R_o criterion leads to a "communication theory" in many ways stronger and more profound than that based on "error probability." To illustrate this point, we prove in Section V that the simplex signal set is optimum for the R_o criterion on the additive White Gaussian noise channel. From our proof, we are able to make some fundamental observations that apply to all signal sets. Finally, in Section VI we show how the R_o criterion leads to the systematic design of "soft decision" demodulators for which an "error probability" criterion would be meaningless.

II. Basic Definitions

We shall suppose the encoded digits X in Fig. 1 are q-ary letters, i.e. that the modulator can generate any of q signals as the transmitted waveform s(t) in each signalling interval. The rate R of the coding systems is the number of binary digits U per encoded digit X (on a long-time basis), i.e. the number of source bits sent per use of the channel. We shall suppose that the demodulated digits Y are q'-ary letters where in general $q' \geq q$. The decoder output digits Û are the estimates of the source delivered to the sink.

From the coding viewpoint, the modulator, waveform channel, and demodulator together constitute a discrete channel with q input letters and q' output letters. We shall assume this channel is memoryless, i.e. that the channel acts independently in each signalling interval, an assumption which is not so restrictive as might first appear since nothing prevents the signalling intervals from being very long (and in fact comprising many intervals from a "smaller" modulator.) In information theory parlance, X and Y constitute the input and output digits respectively of a discrete memoryless channel or (DMC). We shall let $P(y|x)$ denote the probability of receiving the output letter y when the input letter x is sent, and these transition probabilities completely define the DMC.

III. A Sensible Modulation Criterion

The design criterion for modulation systems in almost universal use is "error probability," i.e. $P(y \neq x)$. For this criterion to be meaningful, it is necessary to choose q'=q (usually both are 2) so that the input and output letters can share the same alphabet--a seemingly innocuous step that is fatal to the effective use of coding. The communications engineer who uses this criterion presumes that the purpose of coding is to "correct errors" made by the demodulator, and he commonly refers to codes in general as "error-correcting codes." He somehow believes that if the "error probability" is unacceptably large, then coding should be able to rescue his design and he is inevitably disappointed to learn that the necessary code redundancy is so large that his overall system is quite inefficient.

The next step is to redesign the modulation system to give an acceptable "error probability" without coding.

The crux of the problem is that "error probability"is a modulation design criterion which is sensible only if coding is not used. It should then not be surprising that modulation systems designed on this basis prove to be ill-matched to coding systems.

Our discussion in the previous section suggests that the real goal of the modulation system is to create the "best" DMC as seen by the coding system. We now argue that the sensible design criterion for a digital modulation system is the <u>cut-off rate</u> R_o of this DMC, the bigger R_o for a given average energy E in the signalling interval, the better the modulation system. Mathematically, R_o is given by

$$R_o = -\log_2\{\min_{Q(x)} \sum_y [\sum_x \sqrt{P(y|x)}\, Q(x)]^2\} \tag{1}$$

where the minimization is over all probability distributions Q(x) on the channel input letters.

The true error probability of interest in Fig. 1 is $P_e = P(\hat{U} \neq U)$, i.e. the probability that the bit delivered to the sink differs from that which came from the source. Since the work of Viterbi[1], it has been known that if convolutional coding techniques are used on the DMC, then one can achieve

$$P_e \lesssim c_R 2^{-nR_o} \quad \text{if } R \leq R_o \tag{2}$$

where c_R is a small constant best found by simulation and where n is the constraint length (in channel letters) of the convolutional code. The one number R_o then gives <u>both</u> a region of rates where it is possible to operate with arbitrarily small probability of error <u>and</u> an exponent of error probability (which Viterbi has shown is the best possible exponent for rates R near R_o.) No other single number gives such useful information, not even "channel capacity" which gives only a range of rates where operation is possible. Moreover, R_o is also the "R_{comp}" of sequential decoding[2], i.e. the rate above which the average number of decoding steps per decoded digit becomes infinite. In practice, sequential decoders can comfortably operate at rates R near R_o (or even slightly greater, but only slightly).

The thesis that R_o is the sensible criterion for modulation system design is not new. Some years ago, Wozencraft and Kennedy[3] made this same proposal but it fell then on deaf ears. The intervening years have supplied results that now let us resurrect their proposal and to demonstrate that not only is it the logically right criterion but also that it can be effectively employed in modulation system design.

IV. Signal Space Considerations

We now suppose that the signal s(t) can be written as a linear combination of N orthonormal functions and hence representable by the vector $\underline{s}$ of its coefficients in this expansion. We presume that r(t) is reduced to its projection $\underline{r}$ on these same functions.

If one first considers a demodulator which merely passes along $\underline{r}$ to the decoder (requiring of course $q'=\infty$), one obtains an R_o which upper bounds that for any practical demodulator. Demodulator design then reduces to selecting a finite q' which does not materially reduce the unquantized R_o. It is easily seen from (1) that the unquantized R_o is given by

$$R_o = -\log_2\{\min_{Q(x)} \sum_{i=1}^{q} \sum_{j=1}^{q} \int_{-\infty}^{+\infty} \sqrt{p_i(\underline{\alpha})p_j(\underline{\alpha})}\, d\underline{\alpha}\, Q(i)Q(j)\} \tag{3}$$

where $p_i(\ )$ is the density function for $\underline{r}$ given that the i-th signal vector, $\underline{s}_i$, is transmitted.

The surprising aspect of (3), which seems not to have been previously noticed, is that the formidable-looking integral which it contains can often be easily evaluated. For instance, when the channel has additive white Gaussian noise (AWGN) so that $\underline{r}=\underline{s}+\underline{n}$ where the components of $\underline{n}$ are statistically independent Gaussian random variables with means 0 and variances $N_o/2$, then (3) reduces to the remarkably simple

$$R_o = -\log_2\{\min_{Q(i)} \sum_{i=1}^{q} \sum_{j=1}^{q} e^{-|\underline{s}_i-\underline{s}_j|^2/4N_o} Q(i)Q(j)\} \tag{4}$$

[My student, R. Johannesson, has recently obtained a similarly simple evaluation of (3) when $\underline{n}$ is a general Gaussian variate and is presently extending the results of this paper to that case, i.e. to the non-white additive Gaussian noise channel.]

With (4), we now have a closed-form expression for the unquantized $(q'=\infty)$ R_o of the AWGN channel which is quite amenable to analysis.

V. Simplex Optimality

It has long been conjectured, but never proved, that for the "error probability" criterion the best set of q signal vectors with average energy E on the AWGN channel is the simplex set. We now <u>prove</u> that for the R_o criterion the simplex is indeed optimal.

We see from (4) that R_o depends only on the differences between signal vectors, so that for a given average energy

$$E = \sum_{i=1}^{q} |\underline{s}_i|^2 Q(i) \tag{5}$$

an optimal signal set will have its centroid at the origin, i.e. we may suppose without loss of optimality that

$$\sum_{i=1}^{q} \underline{s}_i Q(i) = \underline{0}. \tag{6}$$

With some slight manipulation, we find that (5) and (6) imply

$$\sum_{i=1}^{q} \sum_{j=1}^{q} |\underline{s}_i-\underline{s}_j|^2 Q(i)Q(j) = 2E \tag{7}$$

which has a surprising simplicity of its own. [We also note that (7) is unchanged if we exclude the term for j=i in the second summation.]

Now letting

$$b = \sum_{i=1}^{q} Q^2(i) \tag{8}$$

we can rewrite (4) as

$$R_o = -\log_2\{\min_{Q(i)} [b+\sum_{i=1}^{q} \sum_{\substack{j=1 \\ j\neq i}}^{q} e^{-|\underline{s}_i-\underline{s}_j|^2/4N_o} Q(i)Q(j)]\}$$

Now using the convexity of the exponential (and noting that the factors Q(i)Q(j) sum to 1-b) and making use of (7), we obtain

$$R_o \leq -\log_2\{\min_{Q(i)} [b+(1-b)e^{-E/2N_o(1-b)}]\} \quad (9)$$

with equality if and only if $|\underline{s}_i-\underline{s}_j|$ is independent of i and j when $i\neq j$, i.e. if and only if the signal set is a simplex. The minimizing Q(i) in (9) is easily seen to be Q(i)=1/q for all i which gives

$$R_o \leq +\log_2 q - \log_2[1+(q-1)e^{-qE/(q-1)2N_o}] \quad (10)$$

with equality if and only if the signal set is a simplex whose signal vectors are equally likely.

Inequality (1) has some profound implications for digital communications. For the simplex set, (10) hold with equality so we see further that

$$\frac{dR_o}{dE} = 0.72\, N_o\, q/[e^{qE/(q-1)2N_o} + (q-1)] \quad (11)$$

which decreases monotonically with E. Hence signal energy is most effectively used when E/N_o is small (and hence when the "error probability" would be large if we were designing a conventional modulation system.) On the other hand, the derivative in (11) is nearly equal to its maximum value at $\frac{E}{N_o}=0$ in the range where the exponential term is dominated by (q-1) and in this range we have

$$\frac{dR_o}{dE} \simeq .72\, \frac{1}{N_o}. \quad (12)$$

Hence

$$R_o \leq .72\, \frac{E}{N_o} \quad (13)$$

is an upper bound applying to any modulation scheme (regardless of q) for the AWGN channel and holding with near equality for the q-ary simplex in the region

$$\frac{E}{N_o} \leq \log_2 q.$$

This shows that the only reason one might choose a large q (rather than q=2) would be if system constraints required a large value of E/N_o since then a large q is required for operation in the energy-efficient range where (13) holds with near equality.

VI. Demodulator Considerations

If one thing were ever clear, it is certainly that the "error probability" criterion is useless when designing a demodulator to make "soft decisions" (q'>q) rather than "hard decisions" (q'=q). On the other hand, the R_o criterion is natural for guiding the communications engineer in this task as we now proceed to show.

Suppose for simplicity of discussion that binary modulation, i.e. q=2, is to be used. We have already seen how the unquantized (i.e. q'=∞) R_o bounds the performance for any demodulator. In practice, one would examine the R_o attainable with q'=2,4,8, 16,... until one had obtained an R_o sufficiently close to the unquantized R_o to justify a practical choice at this smaller number of quantization levels. To carry out such a design procedure, one needs only to know how, for a given q', to set the quantization levels so as to optimize the resulting R_o. Again this problem is nicely amenable to solution.

For a given q', the modulation system reduces to a DMC with q=2 input letters (which we call 0 and 1) and q' output letters (which we call $y_1, y_2, \ldots y_{q'}$). For a given output letter y_i, we define its likelihood ratio as

$$\lambda(y_i) = \frac{P(y_i|0)}{P(y_i|1)}. \quad (14)$$

For q=2, it is easily seen that the optimizing Q(x) in (1) is always Q(0)=Q(1)=1/2 so that (1) reduces to

$$R_o = 1 - \log_2[1+\sum_{i=1}^{q'} \sqrt{P(y_i|0)P(y_i|1)}]. \quad (15)$$

For a given received vector $\underline{\alpha}$, we define its likelihood ratio as

$$\lambda(\underline{\alpha}) = \frac{p_o(\underline{\alpha})}{p_1(\underline{\alpha})}$$

where $p_o(\underline{\alpha})$ is the density function in effect for $\underline{r}$ when 0 is sent. Without loss of optimality in R_o, one can always first convert $\underline{\alpha}$ to $\lambda(\underline{\alpha})$ and hence consider the demodulator design problem as that of choosing quantization thresholds for the scalar quantity $\lambda(\underline{\alpha})$. Let T be a quantization threshold between regions of $\lambda(\underline{\alpha})$ where y_i and y_j respectively will be the demodulator output. If T is optimally chosen, one must have

$$\frac{\partial R_o}{\partial T} = 0.$$

Applying this condition in (15), one can readily show that

$$T = \sqrt{\lambda(b_i)\lambda(b_j)} \quad (16)$$

is the necessary condition for T to be optimal. It follows that the quantization of $\lambda(\underline{\alpha})$ is optimal if and only if each quantization threshold is the geometric mean of the likelihood ratios for the resulting demodulator output letters whose decision regions the threshold separates.

This result suggests the following algorithm to find an optimal set of thresholds for a given desired number q' of output letters. Let y_i be the output letter if $T_{i-1}<\lambda(\underline{\alpha})\leq T_i$ where of course $T_o = -\infty$ and $T_{q'} = +\infty$. One chooses T_1 arbitrarily which then determines $\lambda(y_1)$. One then finds T_2 so that T_1 satisfies 16. The choice of T_2 then determines $\lambda(y_2)$ and one then finds T_3 so that T_2 satisfies (16). Etc. If this procedure can be completed but $T_{q'-1}$ does not satisfy (16), then T_1 must be increased and the procedure repeated. If the procedure cannot be completed, then T_1 must be decreased and the procedure repeated.

My student, L. Lee, has carried out this algorithm to find the optimum quantization rules on the scalar (AWGN) channel. The following table shows his results for E/N_o=1 (0dB):

q'	R_o	T_1	T_2	T_3	T_4	T_5	T_6	T_7
2	.342	0						
4	.4974	-1.032	0	1.032				
8	.5332	-1.799	-1.075	-0.511	0	0.511	1.075	1.799
∞	.5480							

Table I: Optimum Quantization Thresholds on the Scalar AWGN Channel with Binary Antipodal Signals for E/N_o=1 (dB), assuming $\sqrt{E}$ = 1.

In the example of Table I, the conclusion is that q'=8 signal levels is practically optimum, but using "hard decisions" (q'=2) results in serious loss of optimality (about 2dB).

Our main interest in this section has been to show that the R_o criterion provides the communications engineer with a useful way to approach the elusive problem of designing "soft decision" demodulators, but our example is illustrative of the general fact that rarely will a "hard decision" demodulator be a practically optimum choice. The use of a hard decision demodulator will often cause a loss of R_o which translated into dB may well be on the same order as the gain which a sensible coding system could provide.

VI. Remarks

In the above, we have tried to show that the R_o criterion is the sensible one for design of modulation systems in coded digital communications systems. We have also attempted to demonstrate that this criterion is at least as simple to handle in dealing with signal design questions as is demodulator "error probability" and that, moreover, the R_o criterion offers guidance in such matters as the design of "soft decision" demodulators where the "error probability" criterion is mute.

It may seem curious that in a paper intended to deal with coding we have dealt mainly with modulation philosophy. But it is our belief that until modulation systems are designed on an R_o basis the coding engineer will inevitably be faced with a discrete channel for which coding techniques are largely ineffective. It is paramount to the design of efficient coded systems that the modulation engineer aim at maximizing R_o using as simple a signal set (small q) as possible and a consistently larger q'. The net result will be a system where the "error probability" would be large if the demodulator were then compelled to make "hard decisions." Coding techniques such as sequential decoding[3], Viterbi decoding[2] and threshold decoding[4], all of which employ convolutional codes and make use of the "soft decision" information provided by the demodulator, will then naturally suggest themselves as solutions to the design of the coded portion of the over-all digital communications system.

VII. References

1. A. J. Viterbi, "Error Bounds for Convolutional Codes and an Asymptotically Optimum Decoding Algorithm," IEEE Trans. Info. Th., IT-13, pp. 260-269, April 1967.

2. J. M. Wozencraft and I. M. Jacobs, Principles of Communication Engineering, John Wiley and Sons, New York, 1967.

3. J. M. Wozencraft and R. S. Kennedy, "Modulation and Demodulation for Probabilistic Coding," IEEE Trans. Info. Th., IT-12, pp. 291-297, July 1966.

4. J. L. Massey, Threshold Decoding, M.I.T. Press, Cambridge, Mass., 1963.

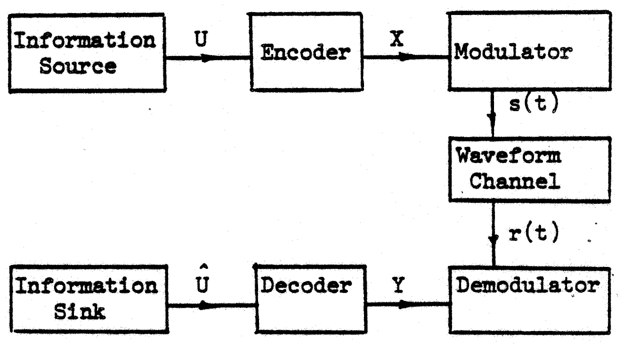

Figure 1. The Basic Information-Theoretic Model of a Digital Communication System