Duality Based Bounds on the Cut-Off Rate of a Discrete-Time Memoryless Rayleigh Fading Channel

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Abstract — **Using a dual expression for the cut-off rate we derive an upper bound on the cut-off rate of an average-power limited discrete-time memoryless Rayleigh fading channel, where neither the transmitter nor the receiver knows the realization of the fading. The bound agrees asymptotically (as the signal-to-noise ratio tends to infinity) with a lower bound that we derive using the primal expression. This establishes that in this regime the cut-off rate is** $\log 2\pi - \gamma - 1 \approx 0.2607$ **nats-per-channel-use away from capacity.**

I. The Channel Model

The complex-valued output $Y \in \mathbb{C}$ of a discrete-time memoryless Rayleigh fading channel is given by

$$Y = Hx + Z, \tag{1}$$

where $x \in \mathbb{C}$ denotes the complex channel input; the complex additive noise $Z \in \mathbb{C}$ is a zero-mean variance-σ^2 circularly-symmetric Gaussian random variable; and the multiplicative noise $H \in \mathbb{C}$ is independent of Z and has a zero-mean unit-variance circularly-symmetric Gaussian distribution.

In this model the non-negative random variable $T = |Y|^2 = |Hx+Z|^2$ is a sufficient statistic for estimating x. Conditionally on x, T has an exponential distribution of mean $|x|^2+\sigma^2$, i.e., it is of density

$$\frac{1}{|x|^2+\sigma^2} e^{-t/(|x|^2+\sigma^2)}, \quad t \geq 0. \tag{2}$$

Using a dual expression for the cut-off rate R_0 we shall study the cut-off rate $R_0(\mathcal{E}_s/\sigma^2)$ of this channel under the average-power constraint

$$E_Q[|X|^2] \leq \mathcal{E}_s. \tag{3}$$

II. The Dual Expression

For any discrete memoryless channel $W(t|x)$ the "primal" expression for the cut-off rate is [1]

$$R_0 = \max E_0(1, Q), \tag{4}$$

where the maximization is over all input distributions satisfying the average-power constraint and where

$$E_0(\varrho, Q) = -\log \sum_t \left(\sum_x Q(x) W(t|x)^{\frac{1}{1+\varrho}} \right)^{\varrho+1}, \quad \varrho \geq 0. \tag{5}$$

In the dual expression [2] R_0 is expressed as

$$R_0 = \max_Q \min_R \left\{ -2 \sum_x Q(x) \log \sum_t \sqrt{W(t|x)R(t)} \right\}, \tag{6}$$

where the minimization is over all probability distributions $R(\cdot)$ on the output alphabet.
Care must be exercised in treating constrained channels [3] or channels with infinite alphabets [1, Sec. 7.3.1].

III. An Upper Bound

Employing (6) with a regularized Gamma output distribution [4] of density

$$\frac{(t+\delta)^{\alpha-1} e^{-\frac{(t+\delta)}{\beta}}}{\beta^\alpha \Gamma(\alpha, \delta/\beta)}, \quad t \geq 0, \quad \alpha, \beta > 0, \delta \geq 0 \tag{7}$$

where $\Gamma(\alpha, \delta/\beta)$ is the incomplete Gamma function, yields the upper bound

$$\begin{aligned} R_0(\mathcal{E}_s/\sigma^2) \leq \log \Gamma(\alpha, \delta/\beta) - 2 \log \Gamma(\frac{\alpha+1}{2}, \frac{\delta}{2\beta} + \frac{\delta}{2\sigma^2}) - \\ - (\alpha+1)\log 2 - \frac{\delta}{\mathcal{E}_s+\sigma^2} + \log(1 + \frac{\mathcal{E}_s+\sigma^2}{\beta}) + \\ + \alpha \log(1 + \frac{\beta+\mathcal{E}_s}{\sigma^2}), \quad \alpha, \beta > 0, \delta \geq 0. \end{aligned} \tag{8}$$

By optimizing over the parameters α, β, and δ we obtain the asymptotic upper bound

$$\lim_{\mathcal{E}_s/\sigma^2 \to \infty} \{ R_0(\mathcal{E}_s/\sigma^2) - \log\log(\mathcal{E}_s/\sigma^2) \} \leq -\log(2\pi). \tag{9}$$

IV. An Asymptotic Lower Bound

Computing $E_0(1, Q)$, for the input distribution Q under which $\log|X|^2$ is uniformly distributed between $\log\log\mathcal{E}_s$ and $\log\mathcal{E}_s$ yields the asymptotic lower bound:

$$\lim_{\mathcal{E}_s/\sigma^2 \to \infty} \{ R_0(\mathcal{E}_s/\sigma^2) - \log\log(\mathcal{E}_s/\sigma^2) \} \geq -\log(2\pi). \tag{10}$$

V. Conclusions

The dual expression can be a useful tool in the study of the cut-off rate. In fact, with very little change, one can similarly study the sphere-packing error exponent.

For the Rayleigh fading channel at a high signal-to-noise ratio, the cut-off rate can be very close to channel capacity. By (9)-(10) the former is given by

$$R_0(\mathcal{E}_s/\sigma^2) = \log(1 + \log(1 + \frac{\mathcal{E}_s}{\sigma^2})) - \log(2\pi) + o(1) \tag{11}$$

and the latter by [4]

$$C(\mathcal{E}_s/\sigma^2) = \log(1 + \log(1 + \frac{\mathcal{E}_s}{\sigma^2})) - \gamma - 1 + o(1), \tag{12}$$

where γ is Euler's constant and the o(1) terms tend to zero as $\mathcal{E}_s/\sigma^2$ tends to infinity.

In this regime, the difference between the channel capacity and the cut-off rate is rather small : approximately 0.26 nats or 0.38 bits per channel use.

References

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[4] A.Lapidoth and S.M.Moser, "Capacity bounds via duality with applications to multi-antenna systems on flat fading channels", preprint 2002.