Duality Bounds on the Cut-Off Rate with Applications to Ricean Fading

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Abstract — **We propose to use an expression of Csiszár & Körner's to upper bound Gallager's $E_0(\varrho, Q, r)$ function. We demonstrate this approach by computing the high SNR asymptotic expansion of the computational cut-off rate of the peak- or average-power limited discrete-time memoryless Ricean fading channel with no — or with only partial — side information at the receiver.**

I. The Cut-Off Rate

Consider a discrete-time memoryless channel over the input alphabet $\mathcal{X}$ and the output alphabet $\mathcal{Y}$. For any input $x \in \mathcal{X}$ let $w(\cdot|x)$ be the density, with respect to some fixed measure μ on $\mathcal{Y}$, of the output distribution that is induced by the input $x \in \mathcal{X}$. Let $g : \mathcal{X} \to [0, \infty)$ be some given cost function, and let the allowed cost $\Upsilon \geq 0$ be fixed.

Following Gallager [1] we define for any probability measure Q on $\mathcal{X}$, for any $\varrho \geq 0$, and for any $r \geq 0$

$$E_0(\varrho, Q, r) = -\log \int \left(\int e^{r(g(x)-\Upsilon)} w(y|x)^{\frac{1}{1+\varrho}} \, \mathrm{d}Q(x) \right)^{\varrho+1} \mathrm{d}\mu(y).$$

We shall say that the cost constraint is *active* if

$$\sup_{Q:\mathsf{E}_Q[g(X)] \leq \Upsilon} E_0(1, Q, 0) < \sup_{Q} E_0(1, Q, 0)$$

in which case we define the cut-off rate $R_0(\Upsilon)$ as:

$$R_0(\Upsilon) = \sup_{Q:\mathsf{E}_Q[g(X)]=\Upsilon} \sup_{r \geq 0} E_0(1, Q, r). \tag{1}$$

II. A Dual Expression

Since the cut-off rate is expressed in (1) as a double maximization, any choice of an input distribution Q and of $r \geq 0$ yields a lower bound on $R_0(\Upsilon)$. To obtain an upper bound on $R_0(\Upsilon)$ we propose to extend to infinite alphabets the dual expression of [2, p.192, ex.23]:

Proposition 1. *Let f_R be a probability density on $\mathcal{Y}$. Then for any input distribution Q on $\mathcal{X}$ satisfying the constraint $\mathsf{E}_Q[g(X)] \geq \Upsilon$, any $\varrho \geq 0$, and any $r \geq 0$*

$$E_0(\varrho, Q, r) \leq -(\varrho+1) \int \log \int w(y|x)^{\frac{1}{1+\varrho}} f_R(y)^{\frac{\varrho}{1+\varrho}} \, \mathrm{d}\mu(y) \, \mathrm{d}Q(x).$$

III. Ricean Fading

For a Ricean fading channel $\mathcal{X} = \mathcal{Y} = \mathbb{C}$, the output density (w.r.t. the Lebesgue measure) corresponding to the input x is

$$w(y|x) = \frac{1}{\pi(\sigma^2 + |x|^2)} e^{-\frac{|y-dx|^2}{\sigma^2+|x|^2}}, \; x, y \in \mathbb{C}$$

where $\sigma^2 > 0$ is the variance of the additive Gaussian noise and $d \in \mathbb{C}$ is the specular component. The cost function is $g(x) = |x|^2$, and $\Upsilon = \mathcal{E}_\mathrm{s}$ is the allowed energy per symbol. Irrespective of whether a peak- or an average-power constraint is imposed we show

$$\lim_{\mathcal{E}_\mathrm{s}/\sigma^2 \uparrow \infty} \left\{ R_0 - \log\log \frac{\mathcal{E}_\mathrm{s}}{\sigma^2} \right\} = \frac{|d|^2}{2} - \log(2\pi) - 2 \log \mathrm{I}_0 \left(\frac{|d|^2}{4} \right)$$

where $\mathrm{I}_0(\cdot)$ is the zero-th order modified Bessel function of the first kind. In the special case of Rayleigh fading, i.e. $d = 0$, the RHS of the above is $-\log 2\pi$, in agreement with [4].

The upper bound is based on Proposition 1 with [3]

$$f_R(y) = \frac{(|y|^2 + \delta)^{\alpha-1} e^{-(|y|^2+\delta)/\beta}}{\pi \beta^\alpha \Gamma(\alpha, \delta/\beta)}, y \in \mathbb{C}, \alpha, \beta > 0, \delta \geq 0$$

where $\Gamma(\cdot, \cdot)$ denotes the incomplete Gamma function. The lower bound is based on choosing $r = 0$ and the peak limited input distribution Q under which X is circularly symmetric with

$$\log |X|^2 \sim \text{Uniform}\left(\log\log \mathcal{E}_\mathrm{s}, \log \mathcal{E}_\mathrm{s}\right).$$

Comparing with channel capacity [3]

$$\lim_{\mathcal{E}_\mathrm{s}/\sigma^2 \uparrow \infty} \left\{ C - \log\log \frac{\mathcal{E}_\mathrm{s}}{\sigma^2} \right\} = \log |d|^2 - 1 - \mathrm{Ei}\left(-|d|^2\right)$$

where $\mathrm{Ei}(\cdot)$ denotes the exponential-integral function, shows that the asymptotic difference between the cut-off rate and the channel capacity for the Ricean fading channel never exceeds 0.73 nats per channel use, irrespective of the specular component d.

The asymptotic expansion can also be extended to the case where the receiver has access to some side information S that is jointly Gaussian with the fading. For $d = 0$ (Rayleigh)

$$\lim_{\mathcal{E}_\mathrm{s}/\sigma^2 \uparrow \infty} \left\{ R_0(\mathcal{E}_\mathrm{s}) - \log\log \frac{\mathcal{E}_\mathrm{s}}{\sigma^2} \right\} = -\log 4\epsilon - \log \mathrm{K}\left(\frac{i}{2} \frac{1-\epsilon^2}{\epsilon} \right)$$

where $\mathrm{K}(\cdot)$ is the complete elliptic integral of the first kind, ϵ^2 is the minimum mean squared-error in estimating the fading from S and $i = \sqrt{-1}$.

References

[1] R. G. Gallager, *Information Theory and Reliable Communication.* John Wiley & Sons, 1968.

[2] I. Csiszár and J. Körner, *Information Theory: Coding Theorems for Discrete Memoryless Systems.* Academic Press, 1981.

[3] A. Lapidoth and S. M. Moser, "Capacity bounds via duality with applications to multiple-antenna systems on flat-fading channels," *IEEE Trans. on Inform. Theory*, vol. 49, pp. 2426–2467, Oct. 2003.

[4] A. Lapidoth and N. Miliou, "Duality based bounds on the cut-off rate of a discrete-time memoryless Rayleigh fading channel," in *Proc. Winter School on Coding and Information Theory*, (Monte Veritá, Ascona, Switzerland), February 24–27, 2003.