On the Fading Number of Multi-Antenna Systems

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Abstract — **It has recently been shown that at high signal-to-noise ratios (SNR) the capacity of multi-antenna systems over flat fading channels (without receiver or transmitter side-information) typically grows only double-logarithmically in the SNR. Here we further refine the analysis and study the "fading number" χ, which we define as the limit of the difference between channel capacity and $\log(1+\log(1+\text{SNR}))$.**

It is suggested that at high SNR, i.e., at rates that significantly exceed the fading number, a capacity increase of one bit per channel use requires the squaring of the SNR, or equivalently, the doubling of the SNR as expressed in decibels. In this loose sense, the fading number can be viewed as the channel limiting rate for power-efficient communication. Note, however, that the fading number may be negative.

While the use of multiple antennae does not typically change the double-logarithmic asymptotic dependence of channel capacity on the SNR, multiple antennae do typically increase the fading number, albeit at times (e.g., in the Rayleigh fading case) only in an additive way that grows only logarithmically with the number of antennae.

I. Channel Model

We consider a channel with n_{T} transmit antennae and n_{R} receive antennae whose time-k output $\mathbf{Y}_k \in \mathbb{C}^{n_{\text{R}}}$ is given by

$$\mathbf{Y}_k = \mathbb{H}_k \mathbf{x}_k + \mathbf{Z}_k = \sum_{t=1}^{n_{\text{T}}} x_k^{(t)} \mathbf{H}_k^{(t)} + \mathbf{Z}_k, \tag{1}$$

where $\mathbf{x}_k = (x_k^{(1)}, \ldots, x_k^{(n_{\text{T}})})^{\mathsf{T}} \in \mathbb{C}^{n_{\text{T}}}$ denotes the time-k input vector, the vectors $\{\mathbf{Z}_k\}$ are spatially and temporally white zero-mean variance-σ^2 circularly-symmetric Gaussians, and the components of the random matrices $\{\mathbb{H}_k\}$ are jointly stationary and ergodic stochastic processes independent of $\{\mathbf{Z}_k\}$. We denote the capacity of this channel under the average power constraint

$$\mathsf{E}\Big[\mathbf{X}_k^{\dagger}\mathbf{X}_k\Big] \le \mathcal{E}_{\text{s}} \tag{2}$$

by $C(\text{SNR})$, where $\text{SNR} = \mathcal{E}_{\text{s}}/\sigma^2$.

Under fairly general conditions the capacity $C(\text{SNR})$ increases at most only double-logarithmically with the SNR. In fact [1]:

Theorem 1. *If $\{\mathbb{H}_k\}$ are IID, $\mathsf{E}\Big[\text{tr}(\mathbb{H}_k^{\dagger}\mathbb{H}_k)\Big] < \infty$, and the differential entropy $h(\mathbb{H}_k)$ exists and is greater than $-\infty$, then*

$$\limsup_{SNR\uparrow\infty} \big\{ C(SNR) - \log\big(1+\log(1+SNR)\big) \big\} < \infty. \tag{3}$$

See [1] for an extension of this result to Gaussian fading with memory.

II. The Fading Number

Motivated by Theorem 1 we define the fading number χ as

$$\chi = \limsup_{\text{SNR}\uparrow\infty} \big\{ C(\text{SNR}) - \log\big(1+\log(1+\text{SNR})\big) \big\}. \tag{4}$$

Thus, whenever χ is finite

$$C = \log\big(1 + e^{\chi}\log(1+\text{SNR})\big) + o(1), \tag{5}$$

where the $o(1)$ term tends to zero as the SNR tends to infinity.

Our interest here is in the calculation of the fading number for multi-antenna systems. While some of the techniques we propose for this calculation are fairly general, here we shall only focus on the case where the fading matrices $\{\mathbb{H}_k\}$ are IID, and the components of $\mathbb{H}_k$ are jointly Gaussian. Dropping time indices, we thus have

$$\mathbb{H} = \mathsf{D} + \tilde{\mathbb{H}},$$

where the mean matrix D is a deterministic $n_{\text{R}} \times n_{\text{T}}$ complex matrix, and where the $n_{\text{T}} \cdot n_{\text{R}}$ components $\{H^{(r,t)}\}_{r,t}$ are zero-mean jointly circularly-symmetric and jointly Gaussian complex random variables with a non-singular covariance matrix.

III. A Lower Bound

A lower bound on the fading number can be obtained by ignoring the additive noise but at the cost of constraining the channel inputs to lie outside the unit circle. More precisely:

Theorem 2. *Let*

$$C^0(\mathcal{E}_s) = \sup I(\mathbf{X}; \mathbb{H}\mathbf{X}), \tag{6}$$

where the supremum is over all probability laws on $\mathbf{X} \in \mathbb{C}^{n_{\text{T}}}$ satisfying (2) and the support constraint

$$\Pr(\mathbf{X}^{\dagger}\mathbf{X} < 0) = 0. \tag{7}$$

Then the fading number χ (4) can be lower bounded by

$$\chi \ge \limsup_{\mathcal{E}_s\uparrow\infty} \Big\{ C^0(\mathcal{E}_s) - \log\big(1+\log(1+\mathcal{E}_s)\big) \Big\}. \tag{8}$$

IV. An Upper Bound

An upper bound on the fading number can be derived from the asymptotic behaviour of any upper bound to the channel capacity. Employing the techniques of [1] one can establish

[1] This research was conducted in part while A. Lapidoth was a Visiting Scholar at the Rockefeller Foundation Bellagio Study & Conference Center, Bellagio, Italy.

that for any non-singular deterministic $n_{\mathrm{R}} \times n_{\mathrm{R}}$ complex matrix A,

$$\chi \leq -m - \log\Gamma(m) - \log|\det(\mathsf{A})|^2 + \sup_{\mathbf{x}\in\mathbb{C}^{n_{\mathrm{T}}}} \Big\{ mE\big[\log\|\mathsf{A}\mathbf{Y}\|^2 | \mathbf{X}=\mathbf{x}\big] - \log\det\Big(E\big[\tilde{\mathbb{H}}\mathbf{x}\mathbf{x}^\dagger\tilde{\mathbb{H}}^\dagger\big] + \sigma^2\mathsf{I}\Big)\Big\},$$

where conditional on $\mathbf{X}=\mathbf{x}$,

$$\mathbf{Y}|\mathbf{X}=\mathbf{x} \sim \mathcal{N}\Big(\mathsf{D}\mathbf{x}, E\big[\tilde{\mathbb{H}}\mathbf{x}\mathbf{x}^\dagger\tilde{\mathbb{H}}^\dagger\big] + \sigma^2\mathsf{I}\Big),$$

and I denotes the $n_{\mathrm{R}} \times n_{\mathrm{R}}$ identity matrix.

V. Some Special Cases

In certain cases the proposed upper and lower bounds on the fading number coincide. This is, for example, the case when the matrix $\mathbb{H}$ is a square matrix, i.e., $n_{\mathrm{T}} = n_{\mathrm{R}} = m$ and has the form

$$\mathbb{H} = d\mathsf{I} + \tilde{\mathbb{H}},$$

where $d \geq 0$ is some deterministic constant, the matrix I denotes the $m \times m$ identity matrix, and the components of $\tilde{\mathbb{H}}$ are independent and identically distributed zero-mean unit-variance complex Gaussians. In this case one can show that

$$\chi = mg_m(d^2) - m - \log\Gamma(m), \tag{9}$$

where the function $g_m(z)$ is defined as

$$g_m(z) = \log(z) - \mathrm{Ei}(-z) + \sum_{j=1}^{m-1}(-1)^j\left(e^{-z}(j-1)! - \frac{(m-1)!}{j(m-1-j)!}\right)z^{-j}$$

and $\mathrm{Ei}(\cdot)$ denotes the exponential integral function

$$\mathrm{Ei}(-\alpha) = -\int_\alpha^\infty \frac{e^{-\xi}}{\xi}\,d\xi, \quad \alpha > 0.$$

As special cases of this result we obtain:

- For the single-antenna Rayleigh fading channel

$$\chi_{\mathrm{Rayleigh}} = -1 - \gamma, \tag{10}$$

  where $\gamma \approx .577$ denotes Euler's constant;
- For the single-antenna Ricean channel:

$$\chi_{\mathrm{Rice}} = -1 + \log d^2 - \mathrm{Ei}(-d^2); \tag{11}$$

- and for the multi-antennae Rayleigh fading channel with m receive antennae:

$$\chi_{m\text{-Rayleigh}} = m\psi(m) - m - \log\Gamma(m), \tag{12}$$

  where $\psi(\xi)$ denotes Euler's psi function so that

$$\psi(m) = -\gamma + \sum_{j=1}^{m-1}\frac{1}{j}.$$

References

[1] A. Lapidoth and S.M. Moser, "Convex-programming bounds on the capacity of flat-fading channels," to be presented at the International Symposium on Information Theory (ISIT'01), Washington DC, June 24–29, 2001.