# Superimposed Coded and Uncoded Transmissions of a Gaussian Source over the Gaussian Channel

Shraga Bross
Department of Electrical Engineering
Technion Israel Institute of Technology
shraga@ee.technion.ac.il

Amos Lapidoth  Stephan Tinguely
Signal and Information Processing Laboratory
Swiss Federal Institute of Technology (ETH) Zurich, Switzerland
{lapidoth, tinguely}@isi.ee.ethz.ch

***Abstract*— We propose to send a Gaussian source over an average-power limited additive white Gaussian noise channel by transmitting a linear combination of the source sequence and the result of its quantization using a high dimensional Gaussian vector quantizer. We show that, irrespective of the rate of the vector quantizer (assumed to be fixed and smaller than the channel's capacity), this transmission scheme is asymptotically optimal (as the quantizer's dimension tends to infinity) under the mean squared-error fidelity criterion. This generalizes the classical result of Goblick about the optimality of scaled uncoded transmission, which corresponds to choosing the rate of the vector quantizer as zero, and the classical source-channel separation approach, which corresponds to choosing the rate of the vector quantizer arbitrarily close to the capacity of the channel.**

## I. Introduction

The minimal distortion with which a memoryless source can be communicated over a memoryless noisy channel is given by the evaluation at channel capacity of the distortion vs. rate function corresponding to the source law and the fidelity criterion [1, Thm. 9.6.3]. For a memoryless Gaussian source and an average-power limited additive white Gaussian noise (AWGN) channel two classical schemes are known to achieve this minimum distortion: the source-channel separation approach [2] and Goblick's "uncoded" scheme [3]. (See also [7], [8], and [9].) Here we shall show that these two schemes can be viewed as the endpoints of a continuum of optimal transmission schemes. In the proposed transmission schemes the transmitted waveform is a linear combination of the source sequence and of the result of its quantization using a Gaussian vector quantizer. The source-channel separation approach corresponds to having the rate of the vector quantizer be arbitrarily close to channel capacity, and Goblick's uncoded scheme corresponds to having the rate of the vector quantizer be zero.

We point out that in contrast to other work on hybrid digital-analog joint source-channel coding, e.g. [4], [5] and [6], we do not aim for issues like "robust" communication, but merely mean to point out a generalization of two well-known optimal schemes. Also, it should be emphasized that our transmission schemes do not increase bandwidth. This should be contrasted with the problem addressed by Shamai, Verdú and Zamir [4] where a memoryless source is to be transmitted to a receiver via *two independent channels* where the transmission over one of the independent channels is required to be uncoded.

## II. Some Definitions

To state our contribution more precisely we need some definitions. The additive white Gaussian noise channel is a channel whose time-$k$ output $Y_k$ takes value in the set of reals $\mathbb{R}$ and is given by

$$Y_k = x_k + Z_k \tag{1}$$

where $x_k \in \mathbb{R}$ denotes the time-$k$ channel input and where the random variables $\{Z_k\}$ are IID, zero-mean, variance-$N$, Gaussian random variables. We say that the length-$n$ sequence of inputs $x_1, \ldots, x_n$ satisfies the average power constraint if

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 \leq P. \tag{2}$$

The capacity of the additive white Gaussian noise channel under the above average power constraint is given by

$$C = \frac{1}{2}\log\left(1 + \frac{P}{N}\right). \tag{3}$$

(We assume throughout that $N$ is strictly larger than zero.)

The memoryless zero-mean variance-$\sigma^2$ source is a source that emits the sequence $\{S_i\}$ of IID zero-mean variance-$\sigma^2$ Gaussian random variables. The variance $\sigma^2$ is assumed to be strictly larger than zero. The distortion vs. rate function $D(R)$ corresponding to this source and to the single-letter squared-error fidelity measure $d(s, \hat{s}) = (s - \hat{s})^2$ is given by

$$D(R) = \sigma^2 2^{-2R}. \tag{4}$$

The evaluation of the above distortion vs. rate function at the capacity of the additive Gaussian noise channel is given by

$$\begin{aligned} D^* &= \sigma^2 2^{-2R}\big|_{R=\frac{1}{2}\log(1+P/N)} \\ &= \sigma^2 \frac{N}{P+N}. \end{aligned} \tag{5}$$

A blocklength-$n$ transmission scheme is a pair of mappings $f_n, \phi_n : \mathbb{R}^n \to \mathbb{R}^n$ with the understanding that when the source emits the sequence $\mathbf{s} \in \mathbb{R}^n$ the sequence

$$f_n(\mathbf{s}) \triangleq \big(x_1(\mathbf{s}), \ldots, x_n(\mathbf{s})\big)$$

is fed to the channel. We require that the transmitted sequence satisfy the average power constraint

$$\mathsf{E}\left[\|f_n(\mathbf{S})\|^2\right] \leq n \cdot P$$

i.e.,

$$\frac{1}{n}\sum_{i=1}^{n}\mathsf{E}\left[x_i^2(\mathbf{S})\right] \leq P. \tag{6}$$

The channel then produces the output sequence $\mathbf{Y}$ whose $i$-th component $Y_i$ is given by $Y_i = x_i(\mathbf{S}) + Z_i, \quad i = 1, \ldots, n$. This output sequence is then mapped by $\phi_n$ to the reconstruction sequence $\hat{\mathbf{S}}$:

$$\begin{aligned}\hat{\mathbf{S}} &= \phi_n(\mathbf{Y}) \\ &\triangleq \big(\hat{S}_1(\mathbf{Y}), \ldots, \hat{S}_n(\mathbf{Y})\big).\end{aligned}$$

The distortion associated with $(f_n, \phi_n)$ is given by

$$d(f_n, \phi_n) = \frac{1}{n}\sum_{i=1}^{n}\mathsf{E}\left[\big(S_i - \hat{S}_i(\mathbf{Y})\big)^2\right] \tag{7}$$

where $S_i$ and $\hat{S}_i$ denote the $i$-th component of $\mathbf{S}$ and $\hat{\mathbf{S}}$ respectively.

A sequence of schemes $\{f_n, \phi_n\}$ indexed by the blocklength $n$ is said to be asymptotically optimal for the transmission of a Gaussian source over the additive white Gaussian noise channel under the mean squared-error fidelity criterion if it results in the transmitted sequence satisfying the average power constraint (6) i.e.,

$$\mathsf{E}\left[\|f_n(\mathbf{S})\|^2\right] \leq nP \tag{8}$$

and if

$$\overline{\lim_{n\to\infty}}\, d(f_n, \phi_n) = D^*. \tag{9}$$

In this submission we propose a sequence of asymptotically optimal transmission schemes parameterized by the free parameter

$$0 < \rho < \frac{1}{2}\log\left(1 + \frac{P}{N}\right) \tag{10}$$

which corresponds to the rate of the Gaussian vector quantizer that we employ. Thus, to each fixed $\rho$ as above, we present a sequence $\{f_n, \phi_n\}$ of coding schemes (parameterized by the blocklength-$n$) that is asymptotically optimal.

## III. The Proposed Scheme

The proposed scheme is conceptually simple, but this simplicity is masked by some of the epsilons and deltas involved. For the sake of clarity and brevity we shall therefore omit these epsilons and deltas here.

At the heart of the scheme is a rate-$\rho$ Gaussian vector quantizer. We denote the quantizer's codebook by $\mathcal{C}$ and assume that its $2^{n\rho}$ codewords are chosen independent of each other, each being drawn uniformly over a centered sphere in $\mathbb{R}^n$. The normalized squared-radius of the sphere is roughly $\sigma^2 - \sigma^2 2^{-2\rho}$ so that the normalized squared-norm of each of the codewords in $\mathcal{C}$ is given roughly by

$$\frac{1}{n}\|\mathbf{u}\|^2 \approx \sigma^2 - \sigma^2 2^{-2\rho} \tag{11}$$

where $n$ denotes the blocklength, $\|\mathbf{u}\|^2$ denotes the sum of the squares of the components of $\mathbf{u}$, and where $\sigma^2$ is the source's variance.

Notice that this would be a rate-$\rho$ optimal vector quantizer for this source and that it would yield a quantization error $\Delta$, where

$$\Delta \approx \sigma^2 2^{-2\rho}. \tag{12}$$

Also, if we slightly increase $\rho$ or slightly decrease the radius of the sphere on which the codewords of the quantizer lie, we could (with very high probability) find a codeword $\mathbf{u}^* \in \mathcal{C}$ such that $\mathbf{s} - \mathbf{u}^*$ would be nearly orthogonal to $\mathbf{u}^*$. Such a codeword $\mathbf{u}^*$ would then satisfy

$$\begin{aligned} n^{-1}\|\mathbf{s} - \mathbf{u}^*\|^2 &\approx n^{-1}\|\mathbf{s}\|^2 - n^{-1}\|\mathbf{u}^*\|^2 \\ &\approx \sigma^2 - \big(\sigma^2 - \sigma^2 2^{-2\rho}\big) \\ &= \sigma^2 2^{-2\rho} \end{aligned} \tag{13}$$

where the first approximation follows because $\mathbf{s} - \mathbf{u}^*$ is nearly orthogonal to $\mathbf{u}^*$ and where the second approximation follows by the law of large numbers for $\|\mathbf{S}\|^2/n$ and from our choice of the radius of the quantizer's sphere.

We can now describe the encoding schemes. Observing the source sequence $\mathbf{s}$, we choose the codeword $\mathbf{u}^*$ in the vector quantizer's codebook $\mathcal{C}$ equiprobably among all codewords that have a "typical" angle to $\mathbf{s}$, i.e. equiprobably among all $\mathbf{u} \in \mathcal{C}$ satisfying

$$\left\langle \frac{\mathbf{s}}{\|\mathbf{s}\|}, \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\rangle \approx \sqrt{1 - 2^{-2\rho}}, \tag{14}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$. If no such $\mathbf{u} \in \mathcal{C}$ exists, the codeword $\mathbf{u}^*$ is choosen to be the all zero sequence $\mathbf{0}$. By slightly shrinking the quantizer's sphere we can guarantee that with very high probability there exists at least one $\mathbf{u} \in \mathcal{C}$ satisfying (14), and consequently, with help of the weak law of large numbers for $\|\mathbf{S}\|/n$, that with very high probability

$$\langle \mathbf{s} - \mathbf{u}^*, \mathbf{u}^* \rangle \approx 0. \tag{15}$$

The transmitted sequence $\mathbf{x} \triangleq f_n(\mathbf{s})$ is now given by a linear combination of $\mathbf{u}^*$ and the source sequence $\mathbf{s}$:

$$\mathbf{x} = f_n(\mathbf{s}) = \alpha\mathbf{s} + \beta\mathbf{u}^* \tag{16}$$

where the coefficients $\alpha = \alpha(\rho)$ and $\beta = \beta(\rho)$ are judiciously chosen as

$$\beta(\rho) = \sqrt{\frac{P+N}{\sigma^2}} - \alpha(\rho), \tag{17}$$

$$\alpha(\rho) = \sqrt{\frac{2^{-2\rho}(N+P) - N}{\sigma^2 2^{-2\rho}}}. \tag{18}$$

This choice of $\alpha$ and $\beta$ is dictated by two requirements. The first is that $\mathbf{x}$ roughly satisfy the power constraint. Indeed, writing

$$\mathbf{x} = (\alpha + \beta)\mathbf{u}^* + \alpha(\mathbf{s} - \mathbf{u}^*) \tag{19}$$

we note that by (15) we shall have $\frac{1}{n}\|\mathbf{x}\|^2 \approx P$ if

$$(\alpha+\beta)^2\|\mathbf{u}\|^2/n + \alpha^2\|\mathbf{s} - \mathbf{u}^*\|^2/n \approx P$$

or, in view of (11) and (13), if

$$(\alpha+\beta)^2\sigma^2\big(1 - 2^{-2\rho}\big) + \alpha^2\sigma^2 2^{-2\rho} \approx P. \tag{20}$$

The second requirement dictating the choice of $\alpha$ and $\beta$ has to do with the decoding and will be described as soon as we describe how the source sequence is reconstructed from the channel output.

This reconstruction takes place in two phases. In the first phase the decoder makes a guess $\hat{\mathbf{u}} \in \mathcal{C}$ of the transmitted codeword $\mathbf{u}^* \in \mathcal{C}$. In the second phase the decoder then makes an estimate $\hat{\mathbf{s}}$ of the source sequence based on $\mathbf{y}$ and $\hat{\mathbf{u}}$. The guess of $\mathbf{u}^*$ in the first phase is based on the observation

$$\mathbf{Y} = (\alpha+\beta)\mathbf{u}^* + \alpha(\mathbf{s}-\mathbf{u}^*) + \mathbf{Z}. \tag{21}$$

The decoder treats the scaled quantization noise $\alpha(\mathbf{s}-\mathbf{u}^*)$ as Gaussian noise, and thus "sees" a signal $((\alpha+\beta)\mathbf{u}^*)$ of average power $(\alpha+\beta)\sigma^2(1-2^{-2\rho})$ contaminated in additive noise $(\alpha(\mathbf{s}-\mathbf{u}^*)+\mathbf{Z})$ of variance $\alpha^2\sigma^2 2^{-2\rho}+N$. Using minimum angle decoding, i.e. $\hat{\mathbf{u}} = \operatorname{argmax}_{\mathbf{u}\in\mathcal{C}} \langle \mathbf{y}, \mathbf{u}\rangle$, it can be shown after some analysis that the decoder will succeed with high probability if [10]

$$\rho < \frac{1}{2}\log\left(1 + \frac{(\alpha+\beta)\sigma^2(1-2^{-2\rho})}{\alpha^2\sigma^2 2^{-2\rho}+N}\right). \tag{22}$$

Replacing this inequality with an (approximate) equality gives us the second condition on $\alpha, \beta$.

In the second phase the reconstructor assumes that the first phase was successful in identifying the codeword $\mathbf{u}^*$. Rearranging terms in (21) we have

$$\frac{\mathbf{Y}-(\alpha+\beta)\mathbf{u}^*}{\alpha} = (\mathbf{s}-\mathbf{u}^*) + \frac{1}{\alpha}\mathbf{Z}.$$

And, since $\mathbf{u}^*$ and $\mathbf{s}-\mathbf{u}^*$ are nearly orthogonal, a reasonable estimator of $\mathbf{S}$ is now the linear estimator

$$\hat{\mathbf{S}} = \mathbf{u}^* + \frac{\alpha^2\Delta}{\alpha^2\Delta+N}\cdot\frac{\mathbf{Y}-(\alpha+\beta)\mathbf{u}^*}{\alpha}, \tag{23}$$

and this is, indeed, the reconstructor we propose. Thus, the reconstruction function $\phi_n$ can be formally defined as

$$\phi_n(\mathbf{y}) = \hat{\mathbf{u}} + \frac{\alpha^2\Delta}{\alpha^2\Delta+N}\cdot\frac{\mathbf{y}-(\alpha+\beta)\hat{\mathbf{u}}}{\alpha}, \tag{24}$$

where $\hat{\mathbf{u}} = \operatorname{argmax}_{\mathbf{u}\in\mathcal{C}}\langle\mathbf{y},\mathbf{u}\rangle$ and $\Delta = \sigma^2 2^{-2\rho}$.

The expected squared error associated to the proposed sequence of schemes $\{f_n, \phi_n\}$

$$d(f_n,\phi_n) = \frac{1}{n}\mathsf{E}\left[\left\|\mathbf{S}-\hat{\mathbf{S}}\right\|^2\right], \tag{25}$$

where the expectation is taken over all $\mathbf{S}$, $\mathbf{Z}$ and $\mathcal{C}$, can now be analyzed by using

$$\mathbf{s}-\hat{\mathbf{s}} \approx \frac{1}{\alpha^2\Delta+N}\left(N\mathbf{s} - \alpha\beta\Delta\mathbf{u}^* - \alpha\Delta\mathbf{z} - (N-\alpha\beta\Delta)\hat{\mathbf{u}}\right),$$

in (25). Writing the expectation as a sum of the individual cross-terms (most of which are straightforwardly bounded) and showing that

$$\mathsf{E}[\langle\mathbf{S},\mathbf{U}^*\rangle] \gtrapprox n\sigma^2(1-2^{-2R}),$$
$$\mathsf{E}\left[\left\langle\mathbf{S},\widehat{\mathbf{U}}\right\rangle\right] \gtrapprox n\sigma^2(1-2^{-2R}),$$
$$\mathsf{E}\left[\left\langle\mathbf{Z},\widehat{\mathbf{U}}\right\rangle\right] \approx 0,$$

then results in

$$\frac{1}{n}\mathsf{E}\left[\left\|\mathbf{S}-\hat{\mathbf{S}}\right\|^2\right] \approx \sigma^2\frac{N}{N+P}.$$

Of course, the rigorous analysis also requires analyzing the effect that the non-existence of a codeword $\mathbf{u}\in\mathcal{C}$ satisfying the encoder condition (14) and the effect that an error in identifying $\mathbf{u}^*$ entail, as well as justifying the approximations that we have presented.

## IV. Conclusion

We have shown that for the transmission of an IID Gaussian source over an AWGN channel with average input power constraint, the minimal expected squared error distortion can be achieved by the superposition of coded and uncoded transmission, for arbitrary power repartition among the schemes. The preserved correlation between the source sequence and the transmitted codeword makes the coded and uncoded schemes perfectly compatible.

## References


[1] R. G. Gallager, *Information Theory and Reliable Communication*, John Wiley & Sons, 1968.

[2] C. E. Shannon, "A Mathematical Theory of Communication", *Bell Systems Technical Journal*, 27, pp. 379-423 and pp. 623-656, July and October 1948.

[3] T. J. Goblick, "Theoretical Limitations on the Transmission of Data from Analog Sources", *IEEE Transactions on Information Theory*, IT-11(4): pp. 558-567, October 1965.

[4] S. Shamai, S. Verdú and R. Zamir, "Systematic Lossy Source/Channel Coding", *IEEE Transactions on Information Theory*, IT-44(2): pp. 564-579, March 1998.

[5] M. Skoglund, N. Phamdo, F. Alajaji, "Designa and Performance of VQ-Based Hybrid Digital-Analog Joint Source-Channel Codes", *IEEE Transactions on Information Theory*, IT-48(3): pp. 708-720, March 2002.

[6] U. Mittal, N. Phamdo, "Hybrid Digital-Analog (HDA) Joint Source-Channel Codes for Broadcasting in Robust Communications", *IEEE Transactions on Information Theory*, IT-48(5): pp. 1082-1102, May 2002.

[7] P. Elias, "Networks of Gaussian Channels with Applications to Feedback Systems", *IEEE Transactions on Information Theory*, IT-13(3): pp. 493-501, July 1967.

[8] U. Mittal and N. Phamdo, "Duality Theorems for Joint Source Channel Coding", *IEEE Transactions on Information Theory*, IT-46(4): pp. 1263-1275, July 2000.

[9] M. Gastpar, B. Rimoldi, and M. Vetterli, "To code, or not to code: lossy source-channel communication revisited," *IEEE Trans. on Inform. Theory*, IT-49, No. 5, pp. 1147–1158, May 2003.

[10] A. Lapidoth, "Nearest-neighbor decoding for additive non-Gaussian noise channels," *IEEE Transactions on Information Theory*, vol. IT-42, pp. 1520-1529, Sept. 1996.