Sending a Bivariate Gaussian Source over a Gaussian MAC with Feedback

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***Abstract*— We consider the problem of transmitting a bivariate Gaussian source over a two-user additive Gaussian multiple-access channel with feedback. Each of the transmitters observes one of the source components and tries to describe it to the common receiver. We are interested in the minimal mean squared error at which the receiver can reconstruct each of the source components.**

In the "symmetric case" we show that, below a certain signal-to-noise ratio threshold which is determined by the source correlation, feedback is useless and the minimal distortion is achieved by uncoded transmission. For the general case we give necessary conditions for the achievability of a distortion pair.

I. Introduction

We consider the problem of transmitting a memoryless bivariate Gaussian source over a two-user additive white Gaussian multiple-access channel with perfect causal feedback from the channel output to both transmitters. Each of the transmitters observes, besides the previous channel outputs, one of the source components which it tries to describe to the receiver subject to an average power constraint on its transmitted signal. Based on the channel output, the receiver estimates the two source components. The quality of the estimate is measured in squared-error distortion on each individual component. We seek the achievable distortion pairs.

We show that in the "symmetric case" — where the transmitters are subjected to the same average power constraint and the ratio of the distortions to be achieved is equal to the ratio of the corresponding source variances — there is a threshold signal-to-noise ratio (SNR), determined by the correlation between the source components, below which feedback is useless and the minimal distortion is achieved by uncoded transmission. This result strengthens a previous result of Lapidoth and Tinguely [1] for the same problem but without feedback. For the general case we give necessary conditions for the achievability of a distortion pair.

Related results by Oohama [2] and Wagner et al. [3] only treated the source coding aspect of this problem by solving the Slepian-Wolf lossy version for the bivariate Gaussian source and by Ozarow [4] who only treated the channel coding aspect by computing the capacity region of the Gaussian multiple-access channel with feedback. We shall, however, not rely on these source coding and channel coding results since the separation theorem does not apply to our problem. That feedback is useless in the symmetric case below some threshold SNR is all the more surprising in view of the recent work of Lapidoth and Wigger [5] who showed that feedback, even if noisy, always increases the capacity region of the Gaussian multiple-access channel.

The work of Stephan Tinguely has been partially supported by the Swiss National Science Foundation under Grant 200021-111863/1.

II. Problem Statement

We consider a discrete-time two-user additive white Gaussian multiple-access channel with perfect and causal feedback from the channel output to both transmitters. The two transmitters of the multiple-access channel each observe one component of a memoryless bivariate Gaussian source and try to communicate it to the receiver.

The time-k output of the Gaussian multiple-access channel is given by

$$Y_k = x_{1,k} + x_{2,k} + Z_k, \tag{1}$$

where $x_{1,k} \in \mathbb{R}$ and $x_{2,k} \in \mathbb{R}$ are the symbols sent by the two transmitters, and Z_k is the time-k additive noise term. The terms $\{Z_k\}$ are independent identically distributed (IID) zero-mean variance-N Gaussian random variables that are independent of the source sequence.

The source symbols produced at time k are $(S_{1,k}, S_{2,k})$ where the $\{(S_{1,k}, S_{2,k})\}$ are IID zero-mean Gaussians of covariance

$$\mathsf{K}_{SS} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}, \tag{2}$$

with $\rho \in [-1,1]$, and $0 < \sigma_i^2 < \infty$, $i = 1,2$. The sequence of the first source component $\{S_{1,k}\}$ is observed by Transmitter 1 and the sequence of the second source component $\{S_{2,k}\}$ is observed by Transmitter 2. Based on their source sequence and the feedback observed up to time k, the transmitters produce their respective time-k channel inputs

$$x_{i,k} = f_{i,k}^{(n)}\left(\mathbf{S}_i, Y^{k-1}\right) \qquad i = 1,2,$$

where we have used the shorthand notation $\mathbf{S}_i = (S_{i,1}, \ldots, S_{i,n})$ and $Y^{k-1} = (Y_1, \ldots, Y_{k-1})$, and where

$$f_{i,k}^{(n)} : \mathbb{R}^n \times \mathbb{R}^{k-1} \to \mathbb{R}, \qquad i = 1,2, \quad k = 1, \ldots, n. \tag{3}$$

The transmitted sequences of the two encoders are average-power limited to P_1 and P_2 respectively, i.e.

$$\frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[\left(f_{i,k}^{(n)}(\mathbf{S}_i, Y^{k-1})\right)^2\right] \le P_i, \quad i = 1,2. \tag{4}$$

The decoder estimates the two source sequences based on the channel output $\mathbf{Y} = (Y_1, \dots Y_n)$. These estimates are denoted by $\hat{\mathbf{S}}_1 = \phi_1^{(n)}(\mathbf{Y})$ and $\hat{\mathbf{S}}_2 = \phi_2^{(n)}(\mathbf{Y})$ respectively, where

$$\phi_i^{(n)} : \mathbb{R}^n \to \mathbb{R}^n, \quad i = 1, 2. \tag{5}$$

We are interested in the minimal expected squared-error distortions at which the receiver can reconstruct each of the source sequences.

Definition 1: Given $\sigma_1, \sigma_2 > 0$, $\rho \in [-1, 1]$, $P_1, P_2 > 0$, and $N > 0$ we say that the tuple $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P_1, P_2, N)$ is *achievable* if there exists a sequence of encoding functions $(f_{1,k}^{(n)}, f_{2,k}^{(n)})$ as in (3) and a sequence of reconstruction pairs $(\phi_1^{(n)}, \phi_2^{(n)})$ as in (5) satisfying the average power constraints (4) and resulting in average distortions that fulfill

$$\varlimsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathsf{E}\left[\left(S_{i,k} - \hat{S}_{i,k}\right)^2\right] \le D_i, \quad i = 1, 2,$$

whenever

$$Y_k = f_{1,k}^{(n)}(\mathbf{S}_1, Y^{k-1}) + f_{2,k}^{(n)}(\mathbf{S}_2, Y^{k-1}) + Z_k,$$

for $k = 1, \dots n$, and $\{(S_{1,k}, S_{2,k})\}$ are IID zero-mean bivariate Gaussian vectors of covariance matrix K_{SS} as in (2) and $\{Z_k\}$ are IID zero-mean variance-N random variables that are independent of $\{(S_{1,k}, S_{2,k})\}$.

The problem we address here is, for given σ_1^2, σ_2^2, ρ, N, P_1, P_2, to find the set of pairs (D_1, D_2) such that $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P_1, P_2, N)$ is achievable.

Remark: As in [1, Section III] it can be shown that there is no loss in generality in assuming that the two source components are of equal variance and that the correlation coefficient is non-negative. Hence, for the remainder we shall assume

$$\rho \in [0, 1] \qquad \text{and} \qquad \sigma_1^2 = \sigma_2^2 = \sigma^2.$$

Furthermore, the convexity argument of [1, Section III] applies also to the case with feedback so that for any given σ^2, ρ, and N, the set of all (D_1, D_2, P_1, P_2) such that $(D_1, D_2, \sigma^2, \sigma^2, \rho, P_1, P_2, N)$ is achievable is a convex set.

Of special interest is the "symmetric case" of this problem where both transmitters are subject to equal power constraints, and where we seek to achieve the same distortion on each source component. That is, for some given N and $P_1 = P_2 = P$ we are interested in

$$D^*(\sigma^2, \rho, P, N) \triangleq \inf\{\max\{D_1, D_2\} : (D_1, D_2, \sigma^2, \sigma^2, \rho, P, P, N) \text{ is achievable}\}.$$

III. Main Results

We now present necessary conditions for the achievability of $(D_1, D_2, \sigma^2, \sigma^2, \rho, P_1, P_2, N)$ and show that in the symmetric case if $P/N \le \rho/(1 - \rho^2)$ then the minimal distortion $D^*(\sigma^2, \rho, P, N)$ is achieved by uncoded transmission and feedback is useless. The corresponding proofs will be discussed in Section IV.

Denote by $R_{S_1,S_2}(D_1, D_2)$ the rate-distortion function for the pair (S_1, S_2) when this pair is observed by one common encoder. For (S_1, S_2) jointly Gaussian as in (2) and with $\sigma_1^2 = \sigma_2^2 = \sigma^2$, we have

$$R_{S_1,S_2}(D_1, D_2) = \begin{cases} \frac{1}{2}\log_2^+\left(\frac{\sigma^4(1-\rho^2)}{D_1 D_2}\right) & \text{if } (D_1, D_2) \in \mathscr{D}_a \\ \frac{1}{2}\log_2^+\left(\frac{\sigma^4(1-\rho^2)}{D_1 D_2 - \left(\rho\sigma^2 - \sqrt{(\sigma^2 - D_1)(\sigma^2 - D_2)}\right)^2}\right) & \text{if } (D_1, D_2) \in \mathscr{D}_b \\ \frac{1}{2}\log_2^+\left(\frac{\sigma^2}{D_1}\right) & \text{if } (D_1, D_2) \in \mathscr{D}_c, \end{cases} \tag{6}$$

where $\log_2^+(x) = \max\{0, \log_2(x)\}$ and the regions $\mathscr{D}_a$, $\mathscr{D}_b$ and $\mathscr{D}_c$ are given by

$$\mathscr{D}_a = \left\{D_1 \le \sigma^2(1 - \rho^2), D_2 \le (\sigma^2(1 - \rho^2) - D_1)\frac{\sigma^2}{\sigma^2 - D_1}\right\}$$

$$\mathscr{D}_b = \Big\{0 \le D_1 \le \sigma^2, \\ (\sigma^2(1 - \rho^2) - D_1)\frac{\sigma^2}{\sigma^2 - D_1} \le D_2 \le \sigma^2(1 - \rho^2) + \rho^2 D_1\Big\}$$

$$\mathscr{D}_c = \left\{0 \le D_1 \le \sigma^2, D_2 > \sigma^2(1 - \rho^2) + \rho^2 D_1\right\}.$$

The expression for $R_{S_1,S_2}(D_1, D_2)$ has been derived in [8] and [1] by different approaches.

Further, denote by $R_{S_1|S_2}(D_1)$ the rate-distortion function for S_1, when S_2 is known to both, the encoder and the decoder, and analogously by $R_{S_2|S_1}(D_2)$ the rate-distortion function for S_2, when S_1 is known to both, the encoder and the decoder. For (S_1, S_2) jointly Gaussian as in (2) and with $\sigma_1^2 = \sigma_2^2 = \sigma^2$, we have

$$R_{S_1|S_2}(D_1) = \frac{1}{2}\log_2^+\left(\frac{\sigma^2(1 - \rho^2)}{D_1}\right) \tag{7}$$

$$R_{S_2|S_1}(D_2) = \frac{1}{2}\log_2^+\left(\frac{\sigma^2(1 - \rho^2)}{D_2}\right). \tag{8}$$

Theorem 1: A necessary condition for the achievability of $(D_1, D_2, \sigma^2, \sigma^2, \rho, P_1, P_2, N)$ is that there exists a $\tilde{\rho} \in [0, 1]$ such that

$$R_{S_1,S_2}(D_1, D_2) \le \frac{1}{2}\log_2\left(1 + \frac{P_1 + P_2 + 2\tilde{\rho}\sqrt{P_1 P_2}}{N}\right) \tag{9}$$

$$R_{S_1|S_2}(D_1) \le \frac{1}{2}\log_2\left(1 + \frac{P_1(1 - \tilde{\rho}^2)}{N}\right) \tag{10}$$

$$R_{S_2|S_1}(D_2) \le \frac{1}{2}\log_2\left(1 + \frac{P_2(1 - \tilde{\rho}^2)}{N}\right), \tag{11}$$

where the explicit forms of the rate-distortion functions on the LHS, are given in (6), (7), and (8) respectively.

In the symmetric case, (9) & (6) yield

$$D \geq \begin{cases} \frac{1}{2}\left(\frac{N\sigma^2(1+\rho)}{N+2P(1+\tilde{\rho})} + \sigma^2(1-\rho)\right) & \text{if } \frac{P}{N} \leq \frac{\rho}{1-\rho^2} \\ \sigma^2\sqrt{\frac{N(1-\rho^2)}{N+2P(1+\tilde{\rho})}} & \text{if } \frac{P}{N} > \frac{\rho}{1-\rho^2} \end{cases} \quad (12)$$

and (10) & (7) (or (11) & (8)) yield

$$D \geq \sigma^2 \frac{N(1-\rho^2)}{N + P(1-\tilde{\rho}^2)}. \quad (13)$$

We denote the RHS of (12) by $\xi(\sigma^2, \rho, P, N, \tilde{\rho})$ and the RHS of (13) by $\psi(\sigma^2, \rho, P, N, \tilde{\rho})$.

Corollary 1: In the symmetric case

$$D^*(\sigma^2, \rho, P, N) \geq \min_{0 \leq \tilde{\rho} \leq 1} \max\left\{\xi(\sigma^2, \rho, P, N, \tilde{\rho}), \psi(\sigma^2, \rho, P, N, \tilde{\rho})\right\}.$$

Note: For $P/N \leq \rho^2/(2(1-\rho)(1+2\rho))$ the minimum in Corollary 1 is achieved by $\tilde{\rho} = 1$, and for all larger P/N the minimum is achieved by the $\tilde{\rho}^*$ for which

$$\xi(\sigma^2, \rho, P, N, \tilde{\rho}^*) = \psi(\sigma^2, \rho, P, N, \tilde{\rho}^*).$$

We can now verify that for $P/N = \rho/(1-\rho^2)$ the lower bound on $D^*(\sigma^2, \rho, P, N)$ from Corollary 1 is achieved by uncoded transmission. For $P/N = \rho/(1-\rho^2)$ the minimizing $\tilde{\rho}$ is $\tilde{\rho}^* = \rho$ leading to the bound

$$D^*(\sigma^2, \rho, P, N) \geq \sigma^2(1-\rho). \quad (14)$$

To see that this is achievable by uncoded transmission, note that in the symmetric case, uncoded transmission of the form $x_{i,k} = \sqrt{P/\sigma^2} S_{i,k}$, $i = 1, 2$ results in the distortion

$$D_u \triangleq \sigma^2 \frac{P(1-\rho^2) + N}{2P(1+\rho) + N}, \quad (15)$$

(see [1, Corollary 2]), which, when evaluated at $P/N = \rho/(1-\rho^2)$ yields the RHS of (14). The following theorem extends this result to all $P/N \leq \rho/(1-\rho^2)$.

Theorem 2: In the symmetric case if $P/N \leq \rho/(1-\rho^2)$ we have

$$D^*(\sigma^2, \rho, P, N) = \sigma^2 \frac{P(1-\rho^2) + N}{2P(1+\rho) + N}, \quad (16)$$

i.e. the minimal distortion is achieved by uncoded transmission, and the availability of feedback is useless.

We conclude this section with a result on the high SNR behaviour of the achievable distortion. Using Theorem 1 and by considering a separate source channel coding scheme [3], [4], one can show

Corollary 2: In the symmetric case

$$\lim_{\frac{P}{N} \to \infty} \sqrt{\frac{P}{N}} D^*(\sigma^2, \rho, P, N) = \sigma^2 \frac{\sqrt{1-\rho^2}}{2}. \quad (17)$$

IV. Sketches of Proofs

We shall discuss the proofs of both theorems but with more particularity on the proof of Theorem 2. We do so, because the basic techniques to the proof of Theorem 1 are the same as in [4] and [6, page 15].

A. Proof of Theorem 1

To prove Theorem 1 we shall use the following lemma

Lemma 1: Let the sequences $\{X_{1,k}\}$ and $\{X_{2,k}\}$ satisfy $\sum_{i=1}^{n} \mathsf{E}\left[X_{i,k}^2\right] \leq nP_i$, $i = 1, 2$. Let $Y_k = X_{1,k} + X_{2,k} + Z_k$, where $\{Z_k\}$ are IID zero-mean variance-N Gaussian, and where for every k, Z_k is independent of $(X_{1,k}, X_{2,k})$. Let $\tilde{\rho} \in [0, 1]$ be given by

$$\tilde{\rho} = \frac{\left|\frac{1}{n}\sum_{k=1}^{n} \mathsf{E}[X_{1,k}X_{2,k}]\right|}{\sqrt{\left(\frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[X_{1,k}^2\right]\right)\left(\frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[X_{2,k}^2\right]\right)}}. \quad (18)$$

Then

$$\sum_{k=1}^{n} I(X_{1,k}, X_{2,k}; Y_k) \leq \frac{n}{2}\log_2\left(1 + \frac{P_1 + P_2 + 2\tilde{\rho}\sqrt{P_1 P_2}}{N}\right), \quad (19)$$

$$\sum_{k=1}^{n} I(X_{1,k}; Y_k | X_{2,k}) \leq \frac{n}{2}\log_2\left(1 + \frac{P_1(1-\tilde{\rho}^2)}{N}\right), \quad (20)$$

$$\sum_{k=1}^{n} I(X_{2,k}; Y_k | X_{1,k}) \leq \frac{n}{2}\log_2\left(1 + \frac{P_2(1-\tilde{\rho}^2)}{N}\right). \quad (21)$$

The proof of Lemma 1 follows from the proof of the main result in [4] and is omitted. Theorem 1 can now be proved by showing

$$nR_{S_1,S_2}(D_1, D_2) \leq I(\mathbf{S}_1, \mathbf{S}_2; \mathbf{Y}) \quad (22)$$

$$I(\mathbf{S}_1, \mathbf{S}_2; \mathbf{Y}) \leq \sum_{k=1}^{n} I(X_{1,k}, X_{2,k}; Y_k), \quad (23)$$

$$nR_{S_1|S_2}(D_1) \leq I(\mathbf{S}_1; \mathbf{Y}|\mathbf{S}_2) \quad (24)$$

$$I(\mathbf{S}_1; \mathbf{Y}|\mathbf{S}_2) \leq \sum_{k=1}^{n} I(X_{1,k}; Y_k | X_{2,k}), \quad (25)$$

$$nR_{S_2|S_1}(D_2) \leq I(\mathbf{S}_2; \mathbf{Y}|\mathbf{S}_1) \quad (26)$$

$$I(\mathbf{S}_2; \mathbf{Y}|\mathbf{S}_1) \leq \sum_{k=1}^{n} I(X_{2,k}; Y_k | X_{1,k}), \quad (27)$$

and by then jointly bounding the expressions on the RHS of (23), (25), and (27) by means of Lemma 1. The proofs of (22) – (27) follow along the same lines as the proof of the univariate analog of which the derivations can be found in [6, page 15] (also coarsely stated in [7, equation (8)]). The main ingredients in those derivations are the convexity of the rate-distortion functions and the data-processing inequality. □

B. Proof of Theorem 2

To prove the theorem we need to show that $D^* \geq D_u$ whenever $P/N \leq \rho/(1-\rho^2)$, where D^* is short for $D^*(\sigma^2, \rho, P, N)$. Since the optimal reconstruction is the conditional expectation, it suffices that we show that a contradiction arises from the assumption:

Assumption 1 (Leading to a contradiction): The encoding rules $\{f_{i,k}^{(n)}\}$ satisfy the average power constraints (4) for some $P_1 = P_2 = P$ satisfying $P/N \leq \rho/(1-\rho^2)$ and, when combined with the optimal conditional expectation reconstructors, achieve D^*, where $D^* < D_u$

To show that this assumption leads to a contradiction, let $\{X_{1,k}, X_{2,k}\}$ and $\{Y_k\}$ be the resulting channel inputs and channel outputs when $\{f_{i,k}^{(n)}\}$ are used to describe the source. Let further $\hat{\mathbf{S}}_1 = \mathsf{E}[\mathbf{S}_1|\mathbf{Y}]$ and $\hat{\mathbf{S}}_2 = \mathsf{E}[\mathbf{S}_2|\mathbf{Y}]$.

We focus on the estimation that Transmitter 2 can make for the vector $\mathbf{W} \triangleq \mathbf{S}_1 - \rho\mathbf{S}_2$ using his knowledged of $\mathbf{S}_2$ and (through the feedback link) $\mathbf{Y}$. This vector is the part of $(\mathbf{S}_1, \mathbf{S}_2)$ which is independent of $\mathbf{S}_2$ and hence initially completely unknown to Transmitter 2. However, from the feedback link Transmitter 2 can retrieve information about $\mathbf{W}$. The contradiction we shall obtain will be on the distortion on $\mathbf{W}$ that can be achieved at Transmitter 2. Under Assumption 1, we shall derive contradictory lower and upper bounds on the achievable value for this distortion.

For any estimator $\varphi^{(n)}(\mathbf{S}_2, \mathbf{Y})$ we set

$$D_W(\varphi^{(n)}) \triangleq \frac{1}{n}\mathsf{E}\left[\|\mathbf{W} - \varphi^{(n)}(\mathbf{S}_2, \mathbf{Y})\|^2\right],$$

where $\|\mathbf{v}\|^2 = \sum_{k=1}^n v_k^2$.

1) "Lower Bound" on $D_W(\varphi^{(n)})$: In this subsection we show that

$$\text{Assumption 1} \;\Rightarrow$$
$$D_W(\varphi^{(n)}) > \sigma^2(1-\rho^2)\frac{N}{N+P(1-\rho^2)} \quad \forall \varphi^{(n)}. \tag{28}$$

The main ingredient is the following lemma:

Lemma 2:

$$\text{Assumption 1} \;\Rightarrow$$
$$I(\mathbf{S}_1; \mathbf{Y}|\mathbf{S}_2) < \frac{n}{2}\log_2\left(1 + \frac{P(1-\rho^2)}{N}\right).$$

The proof of Lemma 2 will be discussed in Section IV-C. Inequality (28) will follow from Lemma 2 if

$$D_W(\varphi^{(n)}) \geq \sigma^2(1-\rho^2)2^{-\frac{2}{n}I(\mathbf{S}_1;\mathbf{Y}|\mathbf{S}_2)}. \tag{29}$$

To this end we denote by $R_W(D)$ the rate-distortion function for a source of the law of $\mathbf{W}$. We then have

$$\begin{aligned}
nR_W(D_W(\varphi^{(n)})) &\overset{a)}{\leq} I(\mathbf{W}; \varphi^{(n)}(\mathbf{S}_2, \mathbf{Y})) \\
&\overset{b)}{\leq} I(\mathbf{W}; \mathbf{Y}, \mathbf{S}_2) \\
&= I(\mathbf{S}_1 - \rho\mathbf{S}_2; \mathbf{Y}, \mathbf{S}_2) \\
&= h(\mathbf{S}_1 - \rho\mathbf{S}_2) - h(\mathbf{S}_1 - \rho\mathbf{S}_2|\mathbf{Y}, \mathbf{S}_2) \\
&\overset{c)}{=} h(\mathbf{S}_1 - \rho\mathbf{S}_2|\mathbf{S}_2) - h(\mathbf{S}_1 - \rho\mathbf{S}_2|\mathbf{Y}, \mathbf{S}_2) \\
&= h(\mathbf{S}_1|\mathbf{S}_2) - h(\mathbf{S}_1|\mathbf{Y}, \mathbf{S}_2) \\
&= I(\mathbf{S}_1; \mathbf{Y}|\mathbf{S}_2),
\end{aligned} \tag{30}$$

where inequality a) follows by the data-processing inequality and the convexity of $R_W(\cdot)$. Inequality b) follows by the data-processing inequality, and c) follows since $\mathbf{S}_2$ and $\mathbf{S}_1 - \rho\mathbf{S}_2$ are independent.

Replacing $R_W(D_W(\varphi^{(n)}))$ in (30) by its explicit form gives

$$\frac{n}{2}\log_2\left(\frac{\sigma^2(1-\rho^2)}{D_W(\varphi^{(n)})}\right) \leq I(\mathbf{S}_1; \mathbf{Y}|\mathbf{S}_2).$$

Rewriting this inequality gives (29), which combines with Lemma 2 to prove (28).

2) "Upper Bound" on minimal $D_W(\varphi^{(n)})$: We show that Assumption 1 implies that the estimator

$$\begin{aligned}
\tilde{\varphi}^{(n)}(\mathbf{S}_2, \mathbf{Y}) &= \alpha \cdot \hat{\mathbf{S}}_1 - \beta \cdot \mathbf{S}_2 \\
&= \alpha\mathsf{E}[\mathbf{S}_1|\mathbf{Y}] - \beta\mathbf{S}_2,
\end{aligned}$$

$$\alpha \triangleq (1-\rho)\frac{\sigma^2}{D^*} \quad \text{and} \quad \beta \triangleq (1-\rho)\frac{\sigma^2 - D^*}{D^*}, \tag{31}$$

violates (28). To prove this we use the following two lemmas:

Lemma 3: For any scheme achieving D^* and any $\delta > 0$ there exists an $n_0(\delta)$ such that for all $n \geq n_0(\delta)$ the following three inequalities are satisfied

$$\frac{1}{n}\sum_{i=1}^n \mathsf{E}\left[S_{1,k}\hat{S}_{1,k}\right] \geq \sigma^2 - D^* - \delta, \tag{32}$$

$$\frac{1}{n}\sum_{i=1}^n \mathsf{E}\left[\hat{S}_{1,k}^2\right] \leq \sigma^2 - D^* + \delta, \tag{33}$$

$$\frac{1}{n}\sum_{i=1}^n \mathsf{E}\left[\hat{S}_{1,k}S_{2,k}\right] \leq \sigma^2 - D^* + 2\delta. \tag{34}$$

Lemma 4: For all $P/N \leq \rho/(1-\rho^2)$ we have

$$\alpha(\rho - \beta) \geq 0. \tag{35}$$

The proofs of Lemma 3 and Lemma 4 will be discussed in Section IV-C. We now derive the desired upper bound on $D_W(\tilde{\varphi}^{(n)})$

$$\begin{aligned}
D_W(\tilde{\varphi}^{(n)}) &= \frac{1}{n}\mathsf{E}\left[\|\mathbf{W} - \tilde{\varphi}(\mathbf{S}_2, \mathbf{Y})\|^2\right] \\
&= \frac{1}{n}\sum_{k=1}^n \mathsf{E}\left[(S_{1,k} - \rho S_{2,k} - \alpha\hat{S}_{1,k} + \beta S_{2,k})^2\right] \\
&= \frac{1}{n}\sum_{k=1}^n \mathsf{E}\left[(S_{1,k} - \alpha\hat{S}_{1,k} - (\rho-\beta)S_{2,k})^2\right] \\
&= \frac{1}{n}\sum_{k=1}^n \Big(\mathsf{E}\left[S_{1,k}^2\right] - 2\alpha\mathsf{E}\left[S_{1,k}\hat{S}_{1,k}\right] \\
&\qquad - 2(\rho-\beta)\mathsf{E}[S_{1,k}S_{2,k}] + \alpha^2\mathsf{E}\left[\hat{S}_{1,k}^2\right] \\
&\qquad + 2\alpha(\rho-\beta)\mathsf{E}\left[\hat{S}_{1,k}S_{2,k}\right] \\
&\qquad + (\rho-\beta)^2\mathsf{E}\left[S_{2,k}^2\right]\Big) \\
&\leq \sigma^2 - 2\alpha(\sigma^2 - D^* - \delta) - 2(\rho-\beta)\rho\sigma^2 \\
&\quad + \alpha^2(\sigma^2 - D^* + \delta) + 2\alpha(\rho-\beta)(\sigma^2 - D^* + 2\delta) \\
&\quad + (\rho-\beta)^2\sigma^2
\end{aligned} \tag{36}$$

where the last step follows from Lemma 3, using the fact that $\alpha \geq 0$, and using Lemma 4.

Upon letting n tend to infinity, we obtain

$$\begin{aligned}
\varlimsup_{n\to\infty} D_W(\tilde{\varphi}^{(n)}) &\leq \sigma^2 - 2\alpha(\sigma^2 - D^* - \delta) - 2(\rho-\beta)\rho\sigma^2 \\
&\quad + \alpha^2(\sigma^2 - D^* + \delta) \\
&\quad + 2\alpha(\rho-\beta)(\sigma^2 - D^* + 2\delta) + (\rho-\beta)^2\sigma^2.
\end{aligned}$$

But since $\delta > 0$ was arbitrary,

$$\begin{aligned}\varlimsup_{n\to\infty} D_W(\tilde{\varphi}^{(n)}) &\leq \sigma^2 - 2\alpha(\sigma^2 - D^*) - 2(\rho-\beta)\rho\sigma^2 \\ &\quad + \alpha^2(\sigma^2 - D^*) + 2\alpha(\rho-\beta)(\sigma^2 - D^*) \\ &\quad + (\rho-\beta)^2\sigma^2 \\ &\stackrel{a)}{\leq} \sigma^2(1-\rho)\left(2 - \frac{\sigma^2}{D^*}(1-\rho)\right) \\ &\stackrel{b)}{<} \sigma^2(1-\rho)\left(2 - \frac{N+2P(1+\rho)}{N+P(1-\rho^2)}(1-\rho)\right) \\ &= \sigma^2(1-\rho^2)\frac{N}{N+2P(1-\rho^2)},\end{aligned}$$

which contradicts (28). Here, a) follows from (31), and b) since we assumed $D^* < D_u$. □

C. Proofs of Lemmas

To prove Lemma 2 we first notice that the assumption $P/N \leq \rho/(1-\rho^2)$ implies, by (6) & (15), that

$$R_{S_1,S_2}(D_u, D_u) = \frac{1}{2}\log_2\left(1 + \frac{2P(1+\rho)}{N}\right).$$

Hence,

$$\begin{aligned}\frac{n}{2}\log_2\left(1 + \frac{2P(1+\rho)}{N}\right) &= nR_{S_1,S_2}(D_u, D_u) \\ &\stackrel{a)}{<} nR_{S_1,S_2}(D^*, D^*) \\ &\stackrel{b)}{\leq} \sum_{k=1}^{n} I(X_{1,k}, X_{2,k}; Y_k) \\ &\stackrel{c)}{\leq} \frac{n}{2}\log_2\left(1 + \frac{2P(1+\tilde{\rho})}{N}\right)\end{aligned} \tag{37}$$

where $\tilde{\rho}$ is given in (18). Here a) follows from the assumption $D^* < D_u$ and the strict monotonicity of $R_{S_1,S_2}(D,D)$; b) follows from (22) & (23); and c) follows from Lemma 1. From (37) and (18) we conclude that

$$\frac{\left|\frac{1}{n}\sum_{k=1}^{n} \mathsf{E}[X_{1,k}X_{2,k}]\right|}{\sqrt{\left(\frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[X_{1,k}^2\right]\right)\left(\frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[X_{2,k}^2\right]\right)}} > \rho. \tag{38}$$

The lemma now follows from (25), Lemma 1 inequality (20), and (38). □

We turn to Lemma 3 and begin by proving Inequalities (32) and (33). By the definition of achievability, for any scheme achieving D^* and any $\delta > 0$ there must exist an $n_0(\delta)$ such that for all $n \geq n_0(\delta)$

$$D^* - \delta < \frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[(S_{i,k} - \hat{S}_{i,k})^2\right] < D^* + \delta \qquad i = 1, 2. \tag{39}$$

Since, by our assumption that $\hat{\mathbf{S}}_1 = \mathsf{E}[\mathbf{S}_1|\mathbf{Y}]$, the orthogonality principle must be satisfied, we obtain from (39) that

$$\sigma^2 - D^* - \delta \leq \frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[S_{1,k}\hat{S}_{1,k}\right] \leq \sigma^2 - D^* + \delta, \tag{40}$$

and

$$\sigma^2 - D^* - \delta \leq \frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[\hat{S}_{1,k}^2\right] \leq \sigma^2 - D^* + \delta. \tag{41}$$

To prove the inequality (34) we start by observing that any scheme achieving D^* must satisfy

$$D_1 = D_2 = D^*. \tag{42}$$

This follows by a time-sharing argument: assume there would exist a scheme achieving D^* with $D_1 = D^*$ and $D_2 = \tilde{D} < D^*$. Then, by symmetry, there would also exist a scheme achieving D^* with $D_1 = \tilde{D} < D^*$ and $D_2 = D^*$. Time-sharing between those two schemes would give a scheme achieving $1/2(D^* + \tilde{D}) < D^*$ which contradicts the definition of D^*.

Statement (42) implies, in view of (39), that for any scheme achieving D^* and any $\delta > 0$ there must exist an $n_0(\delta)$ such that for all $n \geq n_0(\delta)$

$$\frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[(S_{2,k} - \hat{S}_{1,k})^2\right] \geq \frac{1}{n}\sum_{i=1}^{n} \mathsf{E}\left[(S_{1,k} - \hat{S}_{1,k})^2\right] - 2\delta,$$

which is equivalent to

$$\frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[S_{2,k}\hat{S}_{1,k}\right] \leq \frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[S_{1,k}\hat{S}_{1,k}\right] + \delta. \tag{43}$$

Applying (40) to the RHS of (43) gives

$$\frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[S_{2,k}\hat{S}_{1,k}\right] \leq \sigma^2 - D^* + 2\delta.$$ □

To prove Lemma 4 we notice that α is always positive. Hence, the proof of Lemma 4 merely requires showing $\beta \leq \rho$ whenever $P/N \leq \rho/(1-\rho^2)$. Furthermore, since D^* is certainly non-increasing in P/N, and therefore β is non-decreasing in P/N, it is sufficient to show that $\beta \leq \rho$ for $P/N = \rho/(1-\rho^2)$. And this follows from plugging the lower bound (14) for D^* in the expression for β. □

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