Broadcasting Correlated Gaussians

Shraga Bross
School of Engineering
Bar-Ilan University, Israel
brosss@macs.biu.ac.il

Amos Lapidoth Stephan Tinguely
Signal and Information Processing Laboratory
ETH Zurich, Switzerland
{lapidoth, tinguely}@isi.ee.ethz.ch

***Abstract*— We consider a one-to-two Gaussian broadcasting problem where the transmitter observes a memoryless bi-variate Gaussian source and each receiver wishes to estimate one of the source components. The transmitter describes the source pair by means of an average-power-constrained signal and each receiver observes this signal corrupted by a different additive white Gaussian noise. From its respective observation, Receiver 1 wishes to estimate the first source component and Receiver 2 wishes to estimate the second. We seek to characterize the pairs of expected squared-error distortions that are simultaneously achievable at the two receivers.**

Our result is that below a certain SNR-threshold an "uncoded scheme" that sends a linear combination of the source components is optimal. We present a lower bound on this threshold in terms of the source correlation and the distortion at the receiver with weaker channel noise.

I. Introduction

In the single-user scenario where a memoryless Gaussian source is to be transmitted over an additive white Gaussian noise channel it is well known that the minimal squared-error distortion is achieved by an uncoded scheme (see e.g. [1]). In this paper we show that, below some SNR-threshold, a similar result holds for a one-to-two Gaussian broadcasting problem. In our setup, a transmitter observes a bi-variate Gaussian source which it wishes to describe to two receivers by means of an average-power-constrained signal. Each receiver observes the transmitted signal corrupted by a different additive white Gaussian noise. Receiver 1 wishes to estimate the first source component and Receiver 2 wishes to estimate the second. We seek to characterize the pairs of expected squared-error distortions that are simultaneously achievable at the two receivers.

Our result is that below a certain SNR-threshold an "uncoded scheme" that sends a linear combination of the source components is optimal. The SNR-threshold can be expressed as a function of the source correlation and the distortion at the receiver with weaker channel noise.

Similar results on the optimality of uncoded transmission have recently been established for some Gaussian multiple-access scenarios (see e.g. [2], [3], [4]). Related work on broadcast channels with correlated sources can be found in [5], where sufficient conditions for the lossless transmission of finite alphabet sources are given. The difficulty in broadcasting correlated sources is that the source-channel separation theorem does not hold.

The work of Stephan Tinguely has been partially supported by the Swiss National Science Foundation under Grant 200021-111863/1.

II. Problem Statement

Our setup is illustrated in Figure 1. A memoryless bi-variate

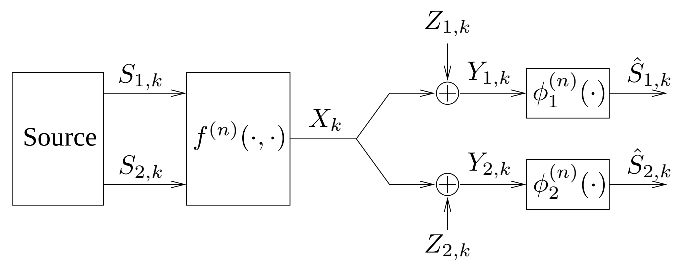

Fig. 1. Two-user Gaussian broadcast channel with bi-variate source.

Gaussian source is combined with a one-to-two Gaussian broadcast channel. The memoryless source emits at each time $k \in \mathbb{Z}$ a bi-variate Gaussian $(S_{1,k}, S_{2,k})$ of zero mean and covariance matrix[1]

$$\mathsf{K}_{SS} = \sigma^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \qquad \text{where} \quad \rho \in [0,1). \tag{1}$$

The source is to be broadcast over a memoryless Gaussian broadcast channel with time-k input $x_k \in \mathbb{R}$, which is subjected to an expected average power constraint

$$\frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[X_k^2\right] \leq P, \tag{2}$$

where $P > 0$ is given. The time-k output $Y_{i,k}$ at Receiver i is given by

$$Y_{i,k} = x_k + Z_{i,k} \qquad i \in \{1,2\},$$

where $Z_{i,k}$ is the time-k additive noise term on the channel to Receiver i. For each $i \in \{1,2\}$ the sequence $\{Z_{i,k}\}_{k=1}^{\infty}$ is independent identically distributed (IID) $\mathcal{N}(0, N_i)$ and independent of the source sequence $\{(S_{1,k}, S_{2,k})\}$, where $\mathcal{N}(\mu, \nu^2)$ denotes the mean-μ variance-ν^2 Gaussian distribution and where we assume[2]

$$N_1 < N_2. \tag{3}$$

For the transmission we consider block encoding schemes where, for block-length n, the transmitted sequence $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is given by

$$\mathbf{X} = f^{(n)}(\mathbf{S}_1, \mathbf{S}_2), \tag{4}$$

[1]The restrictions made on the covariance matrix will be justified in Remark 1 once the problem has been stated completely.

[2]The case $N_1 = N_2$ is equivalent to the problem of sending a bi-variate Gaussian on a single-user Gaussian channel [7].

for some encoding function $f^{(n)}: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, and where we use boldface characters to denote n-tuples, e.g. $\mathbf{S}_1 = (S_{1,1}, S_{1,2}, \ldots, S_{1,n})$. Receiver i's estimate $\hat{\mathbf{S}}_i$ of the source sequence $\mathbf{S}_i$ intended for it, is a function $\phi_i^{(n)}: \mathbb{R}^n \to \mathbb{R}^n$ of its observation $\mathbf{Y}_i$,

$$\hat{\mathbf{S}}_i = \phi_i^{(n)}(\mathbf{Y}_i) \qquad i = 1, 2. \tag{5}$$

The quality of the estimate $\hat{\mathbf{S}}_i$ with respect to the original source sequence $\mathbf{S}_i$ is measured in expected squared-error distortion averaged over the block-length n. We denote this distortion by $\delta_i^{(n)}$, i.e.

$$\delta_i^{(n)} \triangleq \frac{1}{n} \sum_{k=1}^{n} \mathsf{E}\left[(S_{i,k} - \hat{S}_{i,k})^2\right] \qquad i = 1, 2, \tag{6}$$

where $\hat{\mathbf{S}}_i = \phi_i^{(n)}(\mathbf{Y}_i)$ and $\mathbf{Y}_i$. Our interest lies in the set of achievable distortion pairs, where the achievability is defined as follows:

Definition 1 (Achievability): Given $\sigma^2 > 0$, $\rho \in [0, 1)$, $P > 0$ and $0 < N_1 \leq N_2$, we say that the tuple $(D_1, D_2, \sigma^2, \rho, P, N_1, N_2)$ is *achievable* if there exist a sequence of encoding functions $\{f^{(n)}\}$ as in (4) satisfying the average power constraint (2) and sequences of reconstruction functions $\{\phi_1^{(n)}\}$, $\{\phi_2^{(n)}\}$ as in (5) with resulting average distortions that fulfill

$$\varlimsup_{n \to \infty} \delta_i^{(n)} \leq D_i \qquad i = 1, 2,$$

whenever

$$\mathbf{Y}_i = f^{(n)}(\mathbf{S}_1, \mathbf{S}_2) + \mathbf{Z}_i \qquad i = 1, 2, \tag{7}$$

for $\{(S_{1,k}, S_{2,k})\}$ an IID sequence of zero-mean bi-variate Gaussians with covariance matrix as in (1) and $\{Z_{i,k}\}_{k=1}^{\infty}$ IID zero-mean Gaussians of variance N_i, $i = 1, 2$.

The region of all achievable distortion pairs is defined as

Definition 2 ($\mathscr{D}(\sigma^2, \rho, P, N_1, N_2)$): For any σ^2, ρ, P, N_1, and N_2 as in Definition 1 we define $\mathscr{D}(\sigma^2, \rho, P, N_1, N_2)$ (or just $\mathscr{D}$) as the region of all pairs (D_1, D_2) for which $(D_1, D_2, \sigma^2, \rho, P, N_1, N_2)$ is achievable, i.e.

$$\mathscr{D}(\sigma^2, \rho, P, N_1, N_2) = \Big\{(D_1, D_2) : (D_1, D_2, \sigma^2, \rho, P, N_1, N_2) \text{ is achievable}\Big\}.$$

In this paper we study the set $\mathscr{D}$. Before turning to our main results, we state some general properties of the region $\mathscr{D}$. First however, we justify our restriction on the source law made in (1).

Remark 1: For the stated problem, the assumption that the source components are of equal variance and that their correlation coefficient ρ is non-negative incurs no loss in generality. This can be seen by arguments similar to those in [2, Section III]. Furthermore, we exclude the case $\rho = 1$ since for this case the optimality of uncoded transmission at all SNR follows immediately from the corresponding result for the single user scenario [1].

Remark 2: The region $\mathscr{D}$ is closed and convex.
Proof: See [6, Appendix A]. □

To state our second property of $\mathscr{D}$ we need two more definitions. The first one is about the minimal distortion on a single source component.

Definition 3 ($D_{i,\min}$): We say that D_1 is achievable if there exists some D_2 such that $(D_1, D_2) \in \mathscr{D}$. The smallest achievable D_1 is denoted by $D_{1,\min}$. The achievability of D_2 and the distortion $D_{2,\min}$ are analogously defined.

By the classical single-user result

$$D_{i,\min} \triangleq \sigma^2 \frac{N_i}{N_i + P} \qquad i = 1, 2.$$

The second definition is about the boundary points of $\mathscr{D}$.

Definition 4 ($D_1^*(D_2)$ and $D_2^*(D_1)$): For every achievable D_2, we define $D_1^*(D_2)$ as the smallest D_1' such that (D_1', D_2) is achievable, i.e.,

$$D_1^*(D_2) \triangleq \min\left\{D_1' : (D_1', D_2) \in \mathscr{D}\right\}.$$

Similarly,

$$D_2^*(D_1) \triangleq \min\left\{D_2' : (D_1, D_2') \in \mathscr{D}\right\}.$$

Remark 3:

$$D_1^*(D_{2,\min}) = \sigma^2 \frac{N_1 + P(1 - \rho^2)}{N_1 + P}, \tag{8}$$

$$D_2^*(D_{1,\min}) = \sigma^2 \frac{N_2 + P(1 - \rho^2)}{N_2 + P}, \tag{9}$$

with the pair $(D_1^*(D_{2,\min}), D_{2,\min})$ being achievable by setting $X_k = \sqrt{P/\sigma^2} S_{2,k}$, and with the pair $(D_{1,\min}, D_2^*(D_{1,\min}))$ being similarly achievable by setting $X_k = \sqrt{P/\sigma^2} S_{1,k}$.
Proof: See [6, Appendix B]. □

III. Main Results

Our main result states that, below a certain SNR-threshold, every pair $(D_1, D_2) \in \mathscr{D}$ can be achieved by an uncoded scheme, where for every $1 \leq k \leq n$ the transmitted signal is of the form

$$X_k^{\mathrm{u}}(\alpha, \beta) = \sqrt{\frac{P}{\sigma^2(\alpha^2 + 2\alpha\beta\rho + \beta^2)}} \left(\alpha S_{1,k} + \beta S_{2,k}\right), \tag{10}$$

and the corresponding estimate $\hat{S}_{i,k}^{\mathrm{u}}$, $i \in \{1, 2\}$, is the minimum mean squared-error estimate of $S_{i,k}$ based on the scalar observation $Y_{i,k}$, i.e.,

$$\hat{S}_{i,k}^{\mathrm{u}} = \mathsf{E}[S_{i,k} | Y_{i,k}], \qquad i \in \{1, 2\}.$$

We denote the distortions resulting from this scheme by D_1^{u} and D_2^{u}. They are given by

$$D_i^{\mathrm{u}}(\alpha, \beta) = \sigma^2 \frac{\xi_i}{\zeta_i} \qquad i \in \{1, 2\},$$

where

$$\xi_1 = P^2\beta^2(1 - \rho^2) + PN_1(\alpha^2 + 2\alpha\beta\rho + \beta^2(2 - \rho^2)) + N_1^2(\alpha^2 + 2\alpha\beta\rho + \beta^2),$$

$$\xi_2 = P^2\alpha^2(1 - \rho^2) + PN_2(\alpha^2(2 - \rho^2) + 2\alpha\beta\rho + \beta^2) + N_2^2(\alpha^2 + 2\alpha\beta\rho + \beta^2),$$

and

$$\zeta_i = (P+N_i)^2(\alpha^2 + 2\alpha\beta\rho + \beta^2) \qquad i \in \{1,2\}.$$

We shall limit ourselves to transmission schemes with $\alpha, \beta \geq 0$ (because for $\rho \geq 0$, an uncoded transmission scheme with the choice of (α, β) such that $\alpha \cdot \beta < 0$ yields a distortion that is uniformly worse than the choice $(|\alpha|, |\beta|)$.). Consequently, the channel input $X_k^{\mathrm{u}}(\alpha, \beta)$ depends on α, β only via the ratio α/β. Thus, we will sometimes assume, without loss of generality, that $\alpha \in [0,1]$ and $\beta = 1-\alpha$.

We now state our main result.

Theorem 1: For any $(D_1, D_2) \in \mathscr{D}$ and

$$\frac{P}{N_1} \leq \Gamma\left(D_1, \sigma^2, \rho\right), \tag{11}$$

there exist $\alpha^*, \beta^* \geq 0$ such that

$$D_1^{\mathrm{u}}(\alpha^*, \beta^*) \leq D_1 \qquad \text{and} \qquad D_2^{\mathrm{u}}(\alpha^*, \beta^*) \leq D_2,$$

where the threshold Γ is given by

$$\Gamma\left(D_1, \sigma^2, \rho\right) = \begin{cases} \frac{\sigma^4(1-\rho^2) - 2D_1\sigma^2(1-\rho^2) + D_1^2}{D_1(\sigma^2(1-\rho^2) - D_1)} & \text{if} \quad 0 < D_1 < \sigma^2(1-\rho^2), \\ +\infty & \text{otherwise.} \end{cases}$$

Proof: See Section IV. □

For $0 < D_1 < \sigma^2(1-\rho^2)$ the threshold function satisfies $\Gamma \geq 2\rho/(1-\rho)$ with equality for $D_1 = \sigma^2(1-\rho)$. Thus a weaker form of Theorem 1 is

Corollary 1: If

$$\frac{P}{N_1} \leq \frac{2\rho}{1-\rho}, \tag{12}$$

then any $(D_1, D_2) \in \mathscr{D}$ is achievable by the uncoded scheme, i.e. for any $(D_1, D_2) \in \mathscr{D}$ there exist some $\alpha^*, \beta^* \geq 0$ such that

$$D_1^{\mathrm{u}}(\alpha^*, \beta^*) \leq D_1 \qquad \text{and} \qquad D_2^{\mathrm{u}}(\alpha^*, \beta^*) \leq D_2.$$

IV. Proof of Theorem 1

Theorem 1 will now be proved by deriving a lower bound on D_2 as a function of D_1, and verifying that for the uncoded scheme with $D_1^{\mathrm{u}} = D_1$ the corresponding D_2^{u} achieves this lower bound on D_2. To state this lower bound we need two preliminaries. They are given in the following reduction and the following definition.

Reduction 1: It suffices to prove the theorem for pairs (D_1, D_2) where

$$D_1 \leq \sigma^2 \frac{N_1 + P(1-\rho^2)}{N_1 + P}. \tag{13}$$

Proof: By Remark 3

$$D_1^*(D_{2,\min}) = \sigma^2 \frac{N_1 + P(1-\rho^2)}{N_1 + P},$$

so any achievable D_2 allows for a D_1 satisfying (13). □

In view of Reduction 1 we shall assume in the rest of the proof that D_1 satisfies (13). Next, we define by $\tilde{D}_2^*(D_1)$ the minimal distortion that can be achieved on $\mathbf{S}_2$ at Receiver 1 (!) when simultaneously it achieves D_1 on $\mathbf{S}_1$. More precisely:

Definition 5 ($\tilde{D}_2^*(D_1)$): For every $D_1 \geq D_{1,\min}$, we define $\tilde{D}_2^*(D_1)$ as

$$\tilde{D}_2^*(D_1) = \inf\{\tilde{D}_2\},$$

where the infimum is over all families of average-power limited encoders $\{f^{(n)}\}$ and reconstructors $\{\phi_1^{(n)}\}$, $\{\tilde{\phi}_2^{(n)}\}$ with corresponding $\hat{\mathbf{S}}_1 = \phi_1^{(n)}(\mathbf{Y}_1)$ and $\tilde{\mathbf{S}}_2 = \tilde{\phi}_2^{(n)}(\mathbf{Y}_1)$ such that

$$\varlimsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[(S_{1,k} - \hat{S}_{1,k})^2\right] \leq D_1,$$

$$\varlimsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[(S_{2,k} - \tilde{S}_{2,k})^2\right] \leq \tilde{D}_2,$$

where $\tilde{\phi}_2^{(n)}$ is defined analogously to $\phi_1^{(n)}$ as a reconstruction function at Receiver 1, but for $\mathbf{S}_2$ rather than $\mathbf{S}_1$.

Remark 4: The distortion $\tilde{D}_2^*(D_1)$ is the distortion that satisfies the equality

$$R_{S_1,S_2}(D_1, \tilde{D}_2^*(D_1)) = \frac{1}{2}\log_2\left(1 + \frac{P}{N_1}\right), \tag{14}$$

where $R_{S_1,S_2}(\cdot,\cdot)$ denotes the rate-distortion function when the pair S_1, S_2 is observed by a common encoder, i.e.

$$R_{S_1,S_2}(\delta_1, \delta_2) = \min_{\substack{P_{T_1,T_2|S_1,S_2}: \\ \mathsf{E}\left[(S_1-T_1)^2\right] \leq \delta_1 \\ \mathsf{E}\left[(S_2-T_2)^2\right] \leq \delta_2}} I(S_1, S_2; T_1, T_2).$$

We are now ready to prove Theorem 1.

Proof of Theorem 1: The key to the proof is to express the trade-off between the reconstruction fidelity D_1 at Receiver 1 and the reconstruction fidelity D_2 at Receiver 2. This is done in the following lemma.

Lemma 1: If the pair $(D_1, D_2) \in \mathscr{D}$ satisfies (13), and if P/N_1 satisfies (11), then

$$D_2 \geq \Psi(D_1, a_1, a_2), \tag{15}$$

where

$$\Psi(\delta, a_1, a_2) \triangleq \frac{\sigma^2}{P+N_2}\left(\frac{\sigma^2(1-\rho^2)N_1}{\eta(\delta, a_1, a_2)} + N_2 - N_1\right), \tag{16}$$

and where

$$\eta(\delta, a_1, a_2) = \sigma^2 - a_1(\sigma^2 - \delta)(2 - a_1) - a_2\sigma^2(2\rho - a_2) + 2a_1a_2\sqrt{(\sigma^2-\delta)(\sigma^2 - \tilde{D}_2^*)},$$

where we have used the shorthand notation $\tilde{D}_2^*$ for $\tilde{D}_2^*(\delta)$, and where a_1 and a_2 are arbitrary positive real numbers.

Proof: See Section V. □

It now remains to verify that there exist positive a_1, a_2 such that the uncoded scheme achieves the distortion pair $(D_1, \Psi(D_1, a_1, a_2))$. To this end, we need the explicit form of $\tilde{D}_2^*(D_1)$ which, for the cases of interest to us, is given in the following proposition.

Proposition 1: Consider transmitting the bivariate Gaussian source (1) over the AWGN channel that connects the transmitter to Receiver 1. For any D_1 satisfying (13) and P/N_1 satisfying (11), the distortion $\tilde{D}_2^*(D_1)$ is given by

$$\tilde{D}_2^*(D_1) = \sigma^2 \frac{\xi_3}{\zeta_3}, \tag{17}$$

where

$$\begin{aligned}\xi_3 &= P^2\alpha^2(1-\rho^2) + PN_1(\alpha^2(2-\rho^2) + 2\alpha\beta\rho + \beta^2) \\ &\quad + N_1^2(\alpha^2 + 2\alpha\beta\rho + \beta^2), \\ \zeta_3 &= (P+N_1)^2(\alpha^2 + 2\alpha\beta\rho + \beta^2),\end{aligned}$$

where α, β are such that $D_1 = D_1^{\mathrm{u}}(\alpha,\beta)$. Moreover, the pair $(D_1, \tilde{D}_2^*(D_1))$ is achieved by an uncoded scheme with the above choice of α and β.

Proof: See [6, p. 9] □

With Proposition 1 it can now be verified that for every $(D_1, D_2) \in \mathscr{D}$ satisfying (13), and P/N_1 satisfying (11), for the choice of coefficients

$$a_1 = \frac{(\sigma^2 - D_1)\sigma^2 - \rho\sigma^2\sqrt{(\sigma^2-D_1)(\sigma^2-\tilde{D}_2^*(D_1))}}{(\sigma^2-D_1)\tilde{D}_2^*(D_1)},$$

$$a_2 = \frac{\rho\sigma^2 - \sqrt{(\sigma^2-D_1)(\sigma^2-\tilde{D}_2^*(D_1))}}{\tilde{D}_2^*(D_1)},$$

and for the choice of (α,β) so that $D_1^{\mathrm{u}}(\alpha,\beta) = D_1$, the uncoded scheme achieves $D_2^{\mathrm{u}}(\alpha,\beta) = \Psi(D_1, a_1, a_2)$. Notice that for the above choice of a_1, a_2, the bound of Lemma 1 is indeed valid since it can be verified that for all P/N_1 we have $a_1 \geq 0$ and for all P/N_1 satisfying (11) we have $a_2 \geq 0$. □

V. Proof of Lemma 1

Lemma 1 gives a lower bound on the achievable distortion D_2 at Receiver 2 as a function of the distortion D_1 at Receiver 1. In this section we will derive this bound by considering a lower bound for finite block-lengths n and evaluating it in the limit as $n \to \infty$. To ease the evaluation of this limit, we first make a reduction on the coding schemes under consideration.

Reduction 2: To prove Lemma 1 it suffices to consider pairs $(D_1, D_2) \in \mathscr{D}$ that are achievable by coding schemes that achieve D_1 with equality, i.e., for which

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{k=1}^{n} \mathsf{E}\left[(S_{1,k} - \hat{S}_{1,k})^2\right] = D_1, \tag{18}$$

and for which

$$\phi_i^{(n)}(\mathbf{Y}_i) = \mathsf{E}[\mathbf{S}_i|\mathbf{Y}_i] \qquad i \in \{1,2\}. \tag{19}$$

Proof: See [6, Appendix C]. □

We now state our lower bound for finite block-lengths.

Lemma 2: Let a coding scheme $(f^{(n)}, \phi_1^{(n)}, \phi_2^{(n)})$ be given, with $\phi_1^{(n)}$ and $\phi_2^{(n)}$ satisfying (19). Then, for any non-negative coefficients a_1, a_2,

$$\delta_2^{(n)} \geq \Psi(\delta_1^{(n)}, a_1, a_2). \tag{20}$$

Proof: See Section VI. □

With the aid of Lemma 2 and Reduction 2 the proof of Lemma 1 is straightforward:

Proof of Lemma 1: We show that for any non-negative a_1, a_2, the achievable distortion D_2 is lower bounded by

$$D_2 \geq \Psi(D_1, a_1, a_2).$$

By Reduction 2 it suffices to show this for every family of coding schemes $\{f^{(n)}\}$, $\{\phi_1^{(n)}\}$, $\{\phi_2^{(n)}\}$ with $\phi_1^{(n)}$ and $\phi_2^{(n)}$ given in (19) and with associated normalized distortions $\{\delta_1^{(n)}\}$, $\{\delta_2^{(n)}\}$ satisfying

$$\overline{\lim_{n\to\infty}} \delta_1^{(n)} = D_1, \quad \text{and} \quad \overline{\lim_{n\to\infty}} \delta_2^{(n)} \leq D_2, \tag{21}$$

where D_1 satisfies (13). By (21) there exists a subsequence $\{n_k\}$, tending to infinity, such that

$$\lim_{k\to\infty} \delta_1^{(n_k)} = D_1. \tag{22}$$

Hence,

$$\begin{aligned} D_2 &\overset{a)}{\geq} \overline{\lim_{n\to\infty}} \delta_2^{(n)} \\ &\geq \overline{\lim_{k\to\infty}} \delta_2^{(n_k)} \\ &\overset{b)}{\geq} \overline{\lim_{k\to\infty}} \Psi(\delta_1^{(n_k)}, a_1, a_2) \\ &\overset{c)}{=} \Psi(D_1, a_1, a_2), \end{aligned}$$

where $a)$ follows from (21); $b)$ follows from Lemma 2; and $c)$ follows from (22) and from the continuity of $\Psi(\delta, a_1, a_2)$ with respect to δ; a continuity which can be argued as follows. The function $\Psi(\cdot)$ depends on δ only through $\eta(\delta, a_1, a_2)$, and $\eta(\delta, a_1, a_2)$ is on one hand strictly positive for all $P/N_1 > 0$ and all a_1, a_2, and on the other hand continuous in δ because $\tilde{D}_2^*(\delta)$ is continuous in δ. Hence, $\Psi(\cdot)$ is continuous in δ. □

VI. Proof of Lemma 2

The difficulty in proving Lemma 2 lies in relating the two reconstruction fidelities $\delta_1^{(n)}$ and $\delta_2^{(n)}$. One of the problems therein is that for a scheme achieving some $\delta_2^{(n)}$ at Receiver 2, we can only derive bounds on entropy expressions that are conditioned on $\mathbf{S}_2$. However, for a lower bound on $\delta_1^{(n)}$ we would typically like to have a bound on $h(\mathbf{S}_1|\hat{\mathbf{S}}_1)$, without conditioning on $\mathbf{S}_2$. To overcome this difficulty, we furnish Receiver 1 with $\mathbf{S}_2$ as side-information. The proof of Lemma 2 is then obtained by help of the following two lemmas.

Lemma 3: Any scheme achieving some $\delta_2^{(n)}$ at Receiver 2, must produce a $\mathbf{Y}_1$ satisfying

$$I(\mathbf{S}_1; \mathbf{Y}_1|\mathbf{S}_2) \leq \frac{n}{2}\log_2\left(\frac{(P+N_2)\delta_2^{(n)}/\sigma^2 - N_2 + N_1}{N_1}\right). \tag{23}$$

Proof: See [6, Section 5.1]. □

Lemma 4: For any scheme $(f^{(n)}, \phi_1^{(n)}, \phi_2^{(n)})$ satisfying (28) ahead,

$$\frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[\hat{S}_{1,k}S_{2,k}\right] \leq \sqrt{\left(\sigma^2 - \delta_1^{(n)}\right)\left(\sigma^2 - \tilde{D}_2^*(\delta_1^{(n)})\right)}. \tag{24}$$

Proof: See [6, Section 5.2]. □

We are now ready to prove Lemma 2.

Proof of Lemma 2: For this proof, we denote by $\Delta_1^{(n)}$ the least distortion that can be achieved on $\mathbf{S}_1$ at Receiver 1, when $\mathbf{S}_2$ is provided as side-information. The proof will now follow from combining a lower bound on $\delta_2^{(n)}$, as a function of $\Delta_1^{(n)}$, with an upper bound on $\Delta_1^{(n)}$, as a function of $\delta_1^{(n)}$.

We begin with the derivation of the lower bound on $\delta_2^{(n)}$. To this end, let $R_{S_1|S_2}(\cdot)$ denote the rate-distortion function on $\mathbf{S}_1$ when $\mathbf{S}_2$ is given as side-information to both, the encoder and the decoder. Thus, for every $\Delta_1 > 0$,

$$R_{S_1|S_2}(\Delta_1) = \frac{1}{2}\log_2\left(\frac{\sigma^2(1-\rho^2)}{\Delta_1}\right). \tag{25}$$

Since Receiver 1 is connected to the transmitter by a point-to-point link,

$$nR_{S_1|S_2}(\Delta_1^{(n)}) \leq I(\mathbf{S}_1;\mathbf{Y}_1|\mathbf{S}_2). \tag{26}$$

The lower bound on $\delta_2^{(n)}$, in terms of $\Delta_1^{(n)}$, now follows by upper bounding the RHS of (26) by means of Lemma 3, and rewriting the LHS of (26) according to (25). This yields

$$\delta_2^{(n)} \geq \frac{\sigma^2}{P+N_2}\left(\frac{\sigma^2(1-\rho^2)N_1}{\Delta_1^{(n)}} + N_2 - N_1\right). \tag{27}$$

We now turn to the upper bound on $\Delta_1^{(n)}$ in terms of $\delta_1^{(n)}$. Since the RHS of (27) is monotonically decreasing in $\Delta_1^{(n)}$, Lemma 2 will then follow from combining this upper bound on $\Delta_1^{(n)}$ with the lower bound of (27).

The upper bound on $\Delta_1^{(n)}$ follows from analyzing the distortion of a linear estimator of $\mathbf{S}_1$ when Receiver 1 has $\mathbf{S}_2$ as side-information. More precisely, we consider the linear estimator

$$\check{S}_{1,k} = a_1\hat{S}_{1,k} + a_2 S_{2,k}, \qquad k = 1,\ldots,n.$$

To analyze the distortion associated with $\check{\mathbf{S}}_1$, first note that by (19),

$$\mathsf{E}\left[(S_{1,k} - \hat{S}_{1,k})\hat{S}_{1,k}\right] = 0 \qquad \text{for every } 0 \leq k \leq n. \tag{28}$$

Since $\check{\mathbf{S}}_1$ is a valid estimate of $\mathbf{S}_1$ at Receiver 1 when $\mathbf{S}_2$ is given as side-information, we have

$$\begin{aligned}
\Delta_1^{(n)} &\leq \frac{1}{n}\sum_{k=1}^{n}\mathsf{E}\left[(S_{1,k} - \check{S}_{1,k})^2\right] \\
&= \sigma^2 - 2a_1\left(\frac{1}{n}\sum_{k=1}^{n}\mathsf{E}\left[S_{1,k}\hat{S}_{1,k}\right]\right) - 2a_2\rho\sigma^2 \\
&\quad + a_1^2\left(\frac{1}{n}\sum_{k=1}^{n}\mathsf{E}\left[\hat{S}_{1,k}^2\right]\right) \\
&\quad + 2a_1a_2\left(\frac{1}{n}\sum_{k=1}^{n}\mathsf{E}\left[S_{1,k}\hat{S}_{2,k}\right]\right) + a_2^2\sigma^2 \\
&\overset{a)}{=} \sigma^2 - 2a_1(\sigma^2 - \delta_1^{(n)}) - 2a_2\rho\sigma^2 + a_1^2(\sigma^2 - \delta_1^{(n)}) \\
&\quad + 2a_1a_2\left(\frac{1}{n}\sum_{k=1}^{n}\mathsf{E}\left[S_{1,k}\hat{S}_{2,k}\right]\right) + a_2^2\sigma^2, \\
&\leq \sigma^2 - a_1(\sigma^2 - \delta_1^{(n)})(2 - a_1) - a_2\sigma^2(2\rho - a_2) \\
&\quad + 2a_1a_2\sqrt{\left(\sigma^2 - \delta_1^{(n)}\right)\left(\sigma^2 - \tilde{D}_2^*(\delta_1^{(n)})\right)}.
\end{aligned}$$

where in step $a)$ we have used that the normalized summations over $\mathsf{E}\left[\hat{S}_{1,k}^2\right]$ and $\mathsf{E}\left[S_{1,k}\hat{S}_{2,k}\right]$ are both equal to $\sigma^2 - \delta_1^{(n)}$, which follows by (28); and in step $b)$ we have used Lemma 4. Hence, for any $a_1, a_2 \geq 0$,

$$\begin{aligned}
\Delta_1^{(n)} &\leq \sigma^2 - a_1(\sigma^2 - \delta_1^{(n)})(2 - a_1) - a_2\sigma^2(2\rho - a_2) \\
&\quad + 2a_1a_2\sqrt{\left(\sigma^2 - \delta_1^{(n)}\right)\left(\sigma^2 - \tilde{D}_2^*(\delta_1^{(n)})\right)}.
\end{aligned} \tag{29}$$

Denoting the RHS of (29) by $\eta(\delta_1^{(n)}, a_1, a_2)$ and combining (29) with the lower bound (27) gives

$$\delta_2^{(n)} \geq \frac{\sigma^2}{P+N_2}\left(\frac{\sigma^2(1-\rho^2)N_1}{\eta(\delta_1^{(n)}, a_1, a_2)} + N_2 - N_1\right).$$

This is the lower bound of Lemma 2. □

REFERENCES

[1] T. J. Goblick, "Theoretical Limitations on the Transmission of Data from Analog Sources", *IEEE Transactions on Information Theory*, IT-11(4), pp. 558-567, October 1965.

[2] A. Lapidoth, S. Tinguely, "Sending a Bi-Variate Gaussian Source over a Gaussian MAC", in *Proceedings IEEE International Symposium on Information Theory*, Seattle, USA, July 9-14, 2006.

[3] A. Lapidoth, S. Tinguely, "Sending a Bi-Variate Gaussian Source over a Gaussian MAC with Feedback", in *Proceedings IEEE International Symposium on Information Theory*, Nice, France, June 24-29, 2007.

[4] M. Gastpar, "Uncoded transmission is exactly optimal for a simple Gaussian sensor network", in *Proceedings Information Theory and Applications Workshop*, San Diego, CA, USA, January 29 - February 2, 2007.

[5] T S. Han, M. H. M. Costa, "Broadcast Channels with Arbitrarily Correlated Sources", *IEEE Transactions on Information Theory*, IT-33(5), pp. 641-650, September 1987.

[6] S. Bross, A. Lapidoth, S. Tinguely, "Broadcasting Correlated Gaussians", `http://arxiv.org/pdf/0710.5376`, October 2007.

[7] A. Lapidoth, S. Tinguely, "Sending a Bi-Variate Gaussian Source over a Gaussian MAC (with Feedback)", *submitted*.