Gaussian Fading is the Worst Fading

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***Abstract*— The capacity of peak-power limited, single-antenna, non-coherent, flat-fading channels with memory is considered. The emphasis is on the capacity pre-log, i.e., on the limiting ratio of channel capacity to the logarithm of the signal-to-noise ratio (SNR), as the SNR tends to infinity. It is shown that, among all stationary & ergodic fading processes of a given spectral distribution function whose law has no mass point at zero, the Gaussian process gives rise to the smallest pre-log.**

I. Introduction

The subject of this paper is the capacity of peak-power limited, single-antenna, discrete-time, non-coherent, flat-fading channels with memory. The transmitter and receiver are both aware of the law of the fading process, but not of its realization. Our focus is on the capacity in the high signal-to-noise ratio (SNR) regime. Specifically, we study the capacity pre-log, which is defined as the limiting ratio of channel capacity to the logarithm of the SNR, as the SNR tends to infinity. We show that of all stationary & ergodic fading processes of a given spectral distribution function and having no mass point at zero, the Gaussian process yields the smallest pre-log. To state this result precisely we begin with a description of the channel model.

A. Channel Model

We consider a single-antenna flat-fading channel with memory where the time-k channel output $Y_k \in \mathbb{C}$ corresponding to the time-k channel input $x_k \in \mathbb{C}$ is given by

$$Y_k = H_k x_k + Z_k. \tag{1}$$

Here $\mathbb{C}$ denotes the complex field, and the random processes $\{Z_k\}$ and $\{H_k\}$ take value in $\mathbb{C}$ and model the additive and multiplicative noises, respectively. It is assumed that these processes are statistically independent and of a joint law that does not depend on the input sequence $\{x_k\}$.

The additive noise $\{Z_k\}$ is a sequence of independent and identically distributed (IID) zero-mean, variance-σ^2, circularly-symmetric, complex Gaussian random variables. The multiplicative noise ("fading") $\{H_k\}$ is a mean-d, unit-variance, stationary & ergodic stochastic process with spectral distribution function $F(\lambda)$, $-1/2 \le \lambda \le 1/2$, i.e.,

$$\mathsf{E}\left[(H_{k+m}-d)(H_k-d)^*\right] = \int_{-1/2}^{1/2} e^{\mathrm{i}2\pi m\lambda}\,\mathrm{d}F(\lambda) \tag{2}$$

where $\mathrm{i}=\sqrt{-1}$, and where A^* denotes the complex conjugate of A.

B. The Pre-Log

The capacity of our channel under a peak-power constraint A^2 on the input is given by

$$C(\mathrm{SNR}) = \lim_{n\to\infty}\frac{1}{n}\sup I(X_1^n;Y_1^n) \tag{3}$$

where the SNR is defined as

$$\mathrm{SNR} \triangleq \frac{\mathsf{A}^2}{\sigma^2}; \tag{4}$$

A_1^n denotes the sequence $A_1,\ldots,A_n$; and where the maximization is over all joint distributions on $X_1,\ldots,X_n$ satisfying with probability one

$$|X_k|^2 \le \mathsf{A}^2, \qquad k=1,\ldots,n. \tag{5}$$

The capacity *pre-log* is now defined by

$$\Pi \triangleq \varlimsup_{\mathrm{SNR}\to\infty}\frac{C(\mathrm{SNR})}{\log \mathrm{SNR}}. \tag{6}$$

For *Gaussian fading*, i.e., when $\{H_k-d\}$ is a circularly-symmetric, complex Gaussian process, the pre-log Π_{G} is given by the Lebesgue measure of the set of harmonics where the derivative of the spectral distribution function is zero, i.e., [1], [2]

$$\Pi_{\mathrm{G}} = \mu\left(\{\lambda : F'(\lambda)=0\}\right) \tag{7}$$

where $\mu(\cdot)$ denotes the Lebesgue measure on the interval $[-1/2,1/2]$, and where $F'(\cdot)$ denotes the derivative of $F(\cdot)$. (Here the subscript "G" stands for "Gaussian".)

This result indicates that if the fading process is Gaussian and band-limited, then the corresponding channel capacity grows logarithmically in the SNR. Note that otherwise the capacity can increase with the SNR in various ways. For instance, in [3] fading channels are studied that result in a capacity which increases double-logarithmically with the SNR, and in [1] spectral distribution functions are presented for which capacity grows as a fractional power of the logarithm of the SNR.

C. The Main Result

In this paper we show that the Gaussian fading has the lowest pre-log among all fading processes having a given spectral distribution function and having no mass point at zero

$$\Pr[H_k=0]=0, \qquad k\in\mathbb{Z}. \tag{8}$$

Thus, if the stationary & ergodic process $\{H_k\}$ satisfies (8) and is of spectral distribution function $F(\cdot)$ then

$$\begin{aligned}\Pi &\geq \Pi_{\mathrm{G}} \\ &= \mu\left(\{\lambda : F'(\lambda) = 0\}\right). \end{aligned} \tag{9}$$

This is made precise in the following theorem.

Theorem 1: Consider a mean-d, unit-variance, stationary & ergodic fading process $\{H_k\}$ with spectral distribution function $F(\cdot)$ and satisfying

$$\Pr[H_k = 0] = 0, \qquad k \in \mathbb{Z}. \tag{10}$$

Then, the corresponding capacity pre-log Π is lower bounded by

$$\Pi \geq \mu\left(\{\lambda : F'(\lambda) = 0\}\right). \tag{11}$$

To prove this theorem we propose in the next section a lower bound on channel capacity and proceed in Section III to analyze its asymptotic growth as the SNR tends to infinity.

II. A Capacity Lower Bound

To derive a lower bound on the capacity we consider inputs $\{X_k\}$ that are IID, zero-mean, circularly-symmetric, and for which $|X_k|^2$ is uniformly distributed on the interval $\left[0, \mathsf{A}^2\right]$. Our derivation is based on the lower bound

$$I(X_1^n; Y_1^n) \geq I(X_1^n; Y_1^n | H_1^n) - I(H_1^n; Y_1^n | X_1^n) \tag{12}$$

which follows from the chain rule

$$\begin{aligned} I(X_1^n; Y_1^n) &= I(X_1^n, H_1^n; Y_1^n) - I(H_1^n; Y_1^n | X_1^n) \\ &= I(H_1^n; Y_1^n) + I(X_1^n; Y_1^n | H_1^n) - I(H_1^n; Y_1^n | X_1^n) \end{aligned} \tag{13}$$

and the non-negativity of mutual information.

We first study the first term on the right-hand side (RHS) of (12). Making use of the stationarity of the channel and of the fact that the inputs are IID we have

$$\lim_{n\to\infty} \frac{1}{n} I(X_1^n; Y_1^n | H_1^n) = I(X_1; Y_1 | H_1). \tag{14}$$

We now lower bound the RHS of (14) as follows. For any fixed $\Gamma > 0$

$$\begin{aligned} I(X_1; Y_1 | H_1) &= h(H_1 X_1 + Z_1 | H_1) - h(Z_1) \\ &= \int_{|h_1| \geq \Gamma} h(H_1 X_1 + Z_1 | H_1 = h_1)\, \mathrm{d}F_{H_1}(h_1) \\ &\quad + \int_{|h_1| < \Gamma} h(H_1 X_1 + Z_1 | H_1 = h_1)\, \mathrm{d}F_{H_1}(h_1) - h(Z_1) \\ &\geq \int_{|h_1| \geq \Gamma} h(H_1 X_1 + Z_1 | H_1 = h_1)\, \mathrm{d}F_{H_1}(h_1) \\ &\quad + \Pr[|H_1| < \Gamma]\, h(Z_1) - h(Z_1) \\ &\geq \int_{|h_1| \geq \Gamma} \left(\log |h_1|^2 + h(X_1)\right) \mathrm{d}F_{H_1}(h_1) \\ &\quad + \Pr[|H_1| < \Gamma]\, h(Z_1) - h(Z_1) \\ &\geq \Pr[|H_1| \geq \Gamma] \left(\log \Gamma^2 + h(X_1)\right) \\ &\quad + \Pr[|H_1| < \Gamma]\, h(Z_1) - h(Z_1) \\ &= \Pr[|H_1| \geq \Gamma] \left(\log \Gamma^2 + \log \pi + h(|X_1|^2)\right) \\ &\quad + \Pr[|H_1| < \Gamma]\, h(Z_1) - h(Z_1) \\ &= \Pr[|H_1| \geq \Gamma] \log \mathsf{A}^2 + \Pr[|H_1| \geq \Gamma] \log \left(\pi \Gamma^2\right) \\ &\quad + \Pr[|H_1| < \Gamma]\, h(Z_1) - h(Z_1) \\ &= \Pr[|H_1| \geq \Gamma] \log \mathsf{A}^2 + \Pr[|H_1| \geq \Gamma] \log \left(\pi \Gamma^2\right) \\ &\quad + \left(\Pr[|H_1| < \Gamma] - 1\right) \log(\pi e \sigma^2) \\ &= \Pr[|H_1| \geq \Gamma] \log \mathsf{SNR} - \Pr[|H_1| \geq \Gamma] \left(1 - \log \Gamma^2\right) \end{aligned} \tag{15}$$

where $F_{H_1}(\cdot)$ denotes the distribution function of the fading H_1. Here, the first inequality follows by conditioning on X_1; the second by conditioning on Z_1 and by the behavior of differential entropy under scaling; the next inequality because over the range of integration $|h_1| \geq \Gamma$ we have $\log |h_1|^2 \geq \log \Gamma^2$; the subsequent equality follows because X_1 is circularly-symmetric [3, Lemma 6.16]; and the next one follows by computing the entropy of a random variable that is uniformly distributed on the interval $\left[0, \mathsf{A}^2\right]$.

We next turn to the second term on the RHS of (12). In order to upper bound it we proceed along the lines of [4], but for non-Gaussian fading. Let $\mathbf{Y}$, $\mathbf{H}$, and $\mathbf{Z}$ be the respective random vectors $(Y_1, \ldots, Y_n)^{\mathsf{T}}$, $(H_1, \ldots, H_n)^{\mathsf{T}}$, and $(Z_1, \ldots, Z_n)^{\mathsf{T}}$ with $\mathbf{A}^{\mathsf{T}}$ denoting the transpose of $\mathbf{A}$. With this notation, (1) is equivalent to

$$\mathbf{Y} = \mathsf{X}\mathbf{H} + \mathbf{Z} \tag{16}$$

where X is a diagonal matrix with diagonal entries $x_1, \ldots, x_n$. It follows that the covariance matrix of $\mathbf{Y}$ given $x_1, \ldots, x_n$ can be written as

$$\begin{aligned} \mathsf{E}\Big[(\mathbf{Y} - \mathsf{E}[\mathbf{Y}])\, (\mathbf{Y} - \mathsf{E}[\mathbf{Y}])^{\dagger} \;\Big|\; X_1 = x_1, \ldots, X_n = x_n \Big] \\ = \mathsf{X}\mathsf{K}_{\mathbf{HH}}\mathsf{X}^{\dagger} + \sigma^2 \mathsf{I}_n \end{aligned} \tag{17}$$

where I_n is the $n \times n$ identity matrix, $(\cdot)^{\dagger}$ denotes Hermitian conjugation, and

$$\mathsf{K}_{\mathbf{HH}} \triangleq \mathsf{E}\left[(\mathbf{H} - \mathsf{E}[\mathbf{H}])(\mathbf{H} - \mathsf{E}[\mathbf{H}])^{\dagger}\right]. \tag{18}$$

Using the entropy maximizing property of circularly-symmetric Gaussian vectors [5, Theorem 9.6.5], we have

$$\begin{aligned} I(H_1^n; Y_1^n | X_1^n) &= h(Y_1^n | X_1^n) - h(Z_1^n) \\ &\leq \mathsf{E}\left[\log\det\left(\mathsf{I}_n + \frac{1}{\sigma^2}\mathbb{X}\mathsf{K}_{\mathbf{HH}}\mathbb{X}^{\dagger}\right)\right] \\ &\stackrel{(a)}{=} \mathsf{E}\left[\log\det\left(\mathsf{I}_n + \frac{1}{\sigma^2}\mathsf{K}_{\mathbf{HH}}\mathbb{X}^{\dagger}\mathbb{X}\right)\right] \\ &\stackrel{(b)}{\leq} \log\det\left(\mathsf{I}_n + \frac{\mathsf{A}^2}{\sigma^2}\mathsf{K}_{\mathbf{HH}}\right) \\ &= \log\left((\pi e)^n \det\left(\mathsf{I}_n + \mathsf{SNR}\,\mathsf{K}_{\mathbf{HH}}\right)\right) - \log(\pi e)^n \\ &\stackrel{(c)}{=} h(V_1, \ldots, V_n) - \log(\pi e)^n \end{aligned} \tag{19}$$

where $\mathbb{X}$ is a random diagonal matrix with diagonal entries $X_1, \ldots, X_n$; and where $\{V_k\}$ is a zero-mean, stationary & ergodic, circularly-symmetric, complex Gaussian process whose spectral distribution function $F_V(\cdot)$ is given by

$$F_V(\lambda) = \lambda + \mathsf{SNR} F(\lambda), \qquad -1/2 \le \lambda \le 1/2. \tag{20}$$

Here, (a) follows from the identity $\det(\mathsf{I}_n + \mathsf{AB}) = \det(\mathsf{I}_n + \mathsf{BA})$; (b) follows from (5) which implies that $\mathsf{A}^2 \mathsf{I}_n - \mathbb{X}^\dagger \mathbb{X}$ is positive semi-definite with probability one; and (c) follows from the expression of the differential entropy of a circularly-symmetric Gaussian vector and by noting that the covariance matrix of the random vector $(V_1, \ldots, V_n)^\mathsf{T}$ is $\mathsf{I}_n + \mathsf{SNR}\mathsf{K}_{\mathbf{HH}}$. Dividing (19) by n and taking the limit as n tends to infinity yields

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} I(H_1^n; Y_1^n | X_1^n) \\ &\quad \le \lim_{n\to\infty} \frac{1}{n} h(V_1, \ldots, V_n) - \log(\pi e) \\ &\quad = \int_{-1/2}^{1/2} \log\left(1 + \mathsf{SNR} F'(\lambda)\right) \mathrm{d}\lambda \end{aligned} \tag{21}$$

where the equality follows from the expression of the differential entropy rate of a Gaussian process [5, Section 11.5].

Equations (12), (14), (15), and (21) yield the capacity lower bound

$$\begin{aligned} &C(\mathsf{SNR}) \\ &\quad \ge \Pr[|H_1| \ge \Gamma] \log \mathsf{SNR} - \Pr[|H_1| \ge \Gamma] \left(1 - \log \Gamma^2\right) \\ &\qquad - \int_{-1/2}^{1/2} \log\left(1 + \mathsf{SNR} F'(\lambda)\right) \mathrm{d}\lambda \end{aligned} \tag{22}$$

for any fixed $\Gamma > 0$. Note that this lower bound holds for all mean-d, unit-variance, stationary & ergodic fading processes $\{H_k\}$ with spectral distribution function $F(\cdot)$.

III. Asymptotic Analysis

In the following we prove Theorem 1 by computing the limiting ratio of the capacity lower bound (22) to the logarithm of the SNR, as the SNR tends to infinity.

We first show that

$$\lim_{\mathsf{SNR}\to\infty} \int_{-1/2}^{1/2} \frac{\log\left(1 + \mathsf{SNR} F'(\lambda)\right)}{\log \mathsf{SNR}} \mathrm{d}\lambda = \mu\left(\{\lambda : F'(\lambda) > 0\}\right). \tag{23}$$

For this purpose we divide the integral into three parts, depending on whether λ takes part in the set $\mathcal{S}_1$, $\mathcal{S}_2$, or $\mathcal{S}_3$, where

$$\mathcal{S}_1 \triangleq \{\lambda \in [-1/2, 1/2] :\ F'(\lambda) = 0\} \tag{24}$$

$$\mathcal{S}_2 \triangleq \{\lambda \in [-1/2, 1/2] :\ F'(\lambda) \ge 1\} \tag{25}$$

$$\mathcal{S}_3 \triangleq \{\lambda \in [-1/2, 1/2] :\ 0 < F'(\lambda) < 1\}. \tag{26}$$

For $\lambda \in \mathcal{S}_1$ the integrand is zero and hence

$$\lim_{\mathsf{SNR}\to\infty} \int_{\mathcal{S}_1} \frac{\log\left(1 + \mathsf{SNR} F'(\lambda)\right)}{\log \mathsf{SNR}} \mathrm{d}\lambda = 0. \tag{27}$$

For $\lambda \in \mathcal{S}_2$ so that $F'(\lambda) \ge 1$ we note that for sufficiently large SNRs the function

$$\frac{\log\left(1 + \mathsf{SNR} F'(\lambda)\right)}{\log \mathsf{SNR}}$$

is monotonically decreasing in the SNR. Therefore, applying the Monotone Convergence Theorem, we have

$$\begin{aligned} &\lim_{\mathsf{SNR}\to\infty} \int_{\mathcal{S}_2} \frac{\log\left(1 + \mathsf{SNR} F'(\lambda)\right)}{\log \mathsf{SNR}} \mathrm{d}\lambda \\ &\quad = \int_{\mathcal{S}_2} \lim_{\mathsf{SNR}\to\infty} \frac{\log\left(1 + \mathsf{SNR} F'(\lambda)\right)}{\log \mathsf{SNR}} \mathrm{d}\lambda \\ &\quad = \mu\left(\mathcal{S}_2\right) \\ &\quad = \mu\left(\{\lambda : F'(\lambda) \ge 1\}\right). \end{aligned} \tag{28}$$

For $\lambda \in \mathcal{S}_3$ so that $0 < F'(\lambda) < 1$ we have

$$0 \le \frac{\log\left(1 + \mathsf{SNR} F'(\lambda)\right)}{\log \mathsf{SNR}} < \frac{\log(1 + \mathsf{SNR})}{\log \mathsf{SNR}} \le \log(1 + e), \quad \mathsf{SNR} \ge e, \tag{29}$$

where the last inequality follows because for sufficiently large SNRs the function $\log(1 + \mathsf{SNR}) / \log \mathsf{SNR}$ is monotonically decreasing in the SNR. Since $\log(1 + e)$ is integrable over $\mathcal{S}_3$ we can make use of the Dominated Convergence Theorem to obtain

$$\begin{aligned} &\lim_{\mathsf{SNR}\to\infty} \int_{\mathcal{S}_3} \frac{\log\left(1 + \mathsf{SNR} F'(\lambda)\right)}{\log \mathsf{SNR}} \mathrm{d}\lambda \\ &\quad = \int_{\mathcal{S}_3} \lim_{\mathsf{SNR}\to\infty} \frac{\log\left(1 + \mathsf{SNR} F'(\lambda)\right)}{\log \mathsf{SNR}} \mathrm{d}\lambda \\ &\quad = \mu\left(\mathcal{S}_3\right) \\ &\quad = \mu\left(\{\lambda : 0 < F'(\lambda) < 1\}\right). \end{aligned} \tag{30}$$

Adding (27), (28), and (30) yields (23).

To continue with the asymptotic analysis of (22) we now note that by (23)

$$\begin{aligned} \Pi &\triangleq \varlimsup_{\mathsf{SNR}\to\infty} \frac{C(\mathsf{SNR})}{\log \mathsf{SNR}} \\ &\ge \Pr[|H_1| \ge \Gamma] - \mu\left(\{\lambda : F'(\lambda) > 0\}\right) \\ &= \mu\left(\{\lambda : F'(\lambda) = 0\}\right) - \Pr[|H_1| < \Gamma] \end{aligned} \tag{31}$$

for any $\Gamma > 0$. Since, by the theorem's assumption, $\Pr[H_1 = 0] = 0$, the cumulative distribution of $|H_1|$ is continuous at zero so that

$$\lim_{\Gamma \downarrow 0} \Pr[|H_1| < \Gamma] = 0 \tag{32}$$

and Theorem 1 therefore follows from (31) by letting Γ tend to zero from above.

IV. Summary and Conclusion

In this paper we showed that, among all stationary & ergodic fading processes $\{H_k\}$ with spectral distribution function $F(\cdot)$ and satisfying (8), the Gaussian process gives rise to the smallest capacity pre-log. This demonstrates the robustness of the Gaussian assumption in the analysis of fading channels at high SNR.

The result can be extended easily to multiple-input single-output (MISO) fading channels with memory when the fading processes corresponding to the different transmit antennas are independent of each other. An expression for the capacity pre-log for MISO Gaussian fading can be found in [6], [7].

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