Multipath Channels of Bounded Capacity

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***Abstract*— The capacity of discrete-time, non-coherent, multipath fading channels is considered. It is shown that if the delay spread is large in the sense that the variances of the path gains do not decay faster than geometrically, then capacity is bounded in the signal-to-noise ratio.**

I. Introduction

This paper studies non-coherent multipath (frequency-selective) fading channels. Such channels have been investigated extensively in the wideband regime where the signal-to-noise ratio (SNR) is typically small, and it was shown that in the limit as the available bandwidth tends to infinity the capacity of the fading channel is the same as the capacity of the additive white Gaussian noise (AWGN) channel of equal received power, see [1].[1]

When the SNR is large we encounter a different situation. Indeed, it has been shown in [5] for non-coherent *frequency-flat* fading channels that if the fading process is regular in the sense that the present fading cannot be predicted perfectly from its past, then at high SNR capacity only increases double-logarithmically in the SNR. This is in stark contrast to the logarithmic growth of the AWGN capacity. See [6], [7], [8], and [9] for extensions to multi-antenna systems, and see [10] and [11] for extensions to non-regular fading, i.e., when the present fading can be predicted perfectly from its past. Thus, communicating over non-coherent flat-fading channels at high SNR is power inefficient.

In this paper, we show that communicating over non-coherent *multipath* fading channels at high SNR is not merely power inefficient, but even worse: if the delay spread is large in the sense that the variances of the path gains do not decay faster than geometrically, then capacity is *bounded* in the SNR. For such channels, capacity does not tend to infinity as the SNR tends to infinity. To state this result precisely we begin with a mathematical description of the channel model.

A. Channel Model

Let $\mathbb{C}$ and $\mathbb{Z}^+$ denote the set of complex numbers and the set of positive integers, respectively. We consider a discrete-time multipath fading channel whose channel output $Y_k \in \mathbb{C}$ at time $k \in \mathbb{Z}^+$ corresponding to the channel inputs $(x_1, x_2, \ldots, x_k) \in \mathbb{C}^k$ is given by

$$Y_k = \sum_{\ell=1}^{k} H_k^{(k-\ell)} x_\ell + Z_k. \tag{1}$$

Here, $H_k^{(\ell)}$ denotes the time-k gain of the ℓ-th path, and $\{Z_k\}$ is a sequence of independent and identically distributed (IID), zero-mean, variance-σ^2, circularly-symmetric, complex Gaussian random variables. We assume that for each path $\ell \in \mathbb{Z}_0^+$ (with $\mathbb{Z}_0^+$ denoting the set of non-negative integers) the stochastic process $\{H_k^{(\ell)}, k \in \mathbb{Z}^+\}$ is a zero-mean stationary process. We denote its variance and its differential entropy rate by

$$\alpha_\ell \triangleq \mathsf{E}\left[\left|H_k^{(\ell)}\right|^2\right], \qquad \ell \in \mathbb{Z}_0^+$$

and

$$h_\ell \triangleq \lim_{n\to\infty} \frac{1}{n} h\left(H_1^{(\ell)}, H_2^{(\ell)}, \ldots, H_n^{(\ell)}\right), \quad \ell \in \mathbb{Z}_0^+,$$

respectively. We further assume that

$$\sup_{\ell \in \mathbb{Z}_0^+} \alpha_\ell < \infty \qquad \text{and} \qquad \inf_{\ell \in \mathcal{L}} h_\ell > -\infty, \tag{2}$$

where the set $\mathcal{L}$ is defined as $\mathcal{L} \triangleq \{\ell \in \mathbb{Z}_0^+ : \alpha_\ell > 0\}$. We finally assume that the processes

$$\{H_k^{(0)}, k \in \mathbb{Z}^+\}, \{H_k^{(1)}, k \in \mathbb{Z}^+\}, \ldots$$

are independent ("uncorrelated scattering"), that they are jointly independent of $\{Z_k\}$, and that the joint law of

$$\left(\{Z_k\}, \{H_k^{(0)}, k \in \mathbb{Z}^+\}, \{H_k^{(1)}, k \in \mathbb{Z}^+\}, \ldots\right)$$

does not depend on the input sequence $\{x_k\}$. We consider a *non-coherent* channel model where neither the transmitter nor the receiver is cognizant of the realization of $\{H_k^{(\ell)}, k \in \mathbb{Z}^+\}$, $\ell \in \mathbb{Z}_0^+$, but both are aware of their statistic. We do not assume that the path gains are Gaussian.

B. Channel Capacity

Let A_m^n denote the sequence $A_m, A_{m+1}, \ldots, A_n$. We define the *capacity* as

$$C(\text{SNR}) \triangleq \varliminf_{n\to\infty} \frac{1}{n} \sup I\left(X_1^n; Y_1^n\right), \tag{3}$$

where the maximization is over all joint distributions on $X_1, X_2, \ldots, X_n$ satisfying the power constraint

$$\frac{1}{n} \sum_{k=1}^{n} \mathsf{E}\left[|X_k|^2\right] \le \mathrm{P}, \tag{4}$$

[1] However, in contrast to the infinite bandwidth capacity of the AWGN channel where the conditions on the capacity achieving input distribution are not so stringent, the infinite bandwidth capacity of non-coherent fading channels can only be achieved by signaling schemes which are "peaky"; see also [2], [3], [4] and references therein.

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and where SNR is defined as

$$\mathrm{SNR} \triangleq \frac{\mathsf{P}}{\sigma^2}. \tag{5}$$

By Fano's inequality, no rate above $C(\mathrm{SNR})$ is achievable.[2] (See [13] for a definition of an achievable rate.)

Notice that the above channel (1) is generally not stationary[3] since the number of terms (paths) influencing Y_k depends on k. It is therefore *prima facie* not clear whether the liminf on the RHS of (3) is a limit.

C. Main Result

Theorem 1: Consider the above channel model. Then

$$\left(\varliminf_{\ell\to\infty} \frac{\alpha_{\ell+1}}{\alpha_\ell} > 0\right) \implies \left(\sup_{\mathrm{SNR}>0} C(\mathrm{SNR}) < \infty\right), \tag{6}$$

where we define, for any $a > 0$, $a/0 \triangleq \infty$ and $0/0 \triangleq 0$.

For example, when $\{\alpha_\ell\}$ is a geometric sequence, i.e., $\alpha_\ell = \rho^\ell$, $\ell \in \mathbb{Z}_0^+$ for some $0 < \rho < 1$, then capacity is bounded.

Theorem 1 is proved in Section II where it is even shown that (6) would continue to hold if we replaced the liminf in (3) by a limsup. Section III addresses briefly multipath channels of *unbounded* capacity.

II. Proof of Theorem 1

The proof follows along the same lines as the proof of [14, Thm. 1i)].

We first note that it follows from the left-hand side (LHS) of (6) that we can find an $\ell_0 \in \mathbb{Z}_0^+$ and a $0 < \rho < 1$ so that $\alpha_{\ell_0} > 0$ and

$$\frac{\alpha_{\ell+1}}{\alpha_\ell} \geq \rho, \qquad \ell = \ell_0, \ell_0+1, \ldots. \tag{7}$$

We continue with the chain rule for mutual information

$$\frac{1}{n} I(X_1^n; Y_1^n) = \frac{1}{n}\sum_{k=1}^{\ell_0} I\big(X_1^n; Y_k \big| Y_1^{k-1}\big) + \frac{1}{n}\sum_{k=\ell_0+1}^{n} I\big(X_1^n; Y_k \big| Y_1^{k-1}\big). \tag{8}$$

Each term in the first sum on the right-hand side (RHS) of (8) is upper bounded by[4]

$$\begin{aligned} I\big(X_1^n; Y_k\big|Y_1^{k-1}\big) &\leq h(Y_k) - h\Big(Y_k\Big|Y_1^{k-1}, X_1^n, H_k^{(0)}, H_k^{(1)}, \ldots, H_k^{(k-1)}\Big)\\ &\leq \log\left(\pi e\left(\sigma^2 + \sum_{\ell=1}^{k} \alpha_{k-\ell}\mathsf{E}\left[|X_\ell|^2\right]\right)\right) - \log\left(\pi e \sigma^2\right)\\ &\leq \log\left(1 + \sup_{\ell\in\mathbb{Z}_0^+} \alpha_\ell \cdot n \cdot \mathrm{SNR}\right), \end{aligned} \tag{9}$$

where the first inequality follows because conditioning cannot increase entropy; the second inequality follows from the entropy maximizing property of Gaussian random variables [13, Thm. 9.6.5]; and the last inequality follows by upper bounding $\alpha_\ell \leq \sup_{\ell\in\mathbb{Z}_0^+} \alpha_\ell$, $\ell = 0, 1, \ldots, k-1$ and from the power constraint (4).

For $k = \ell_0 + 1, \ell_0 + 2, \ldots, n$, we upper bound $I\big(X_1^n; Y_k\big|Y_1^{k-1}\big)$ using the general upper bound for mutual information [5, Thm. 5.1]

$$I(X;Y) \leq \int D\big(W(\cdot|x)\big\|R(\cdot)\big)\,\mathrm{d}Q(x), \tag{10}$$

where $D(\cdot\|\cdot)$ denotes relative entropy, $W(\cdot|\cdot)$ is the channel law, $Q(\cdot)$ denotes the distribution on the channel input X, and $R(\cdot)$ is any distribution on the output alphabet.[5] Thus, any choice of output distribution $R(\cdot)$ yields an upper bound on the mutual information.

For any given $Y_1^{k-1} = y_1^{k-1}$, we choose the output distribution $R(\cdot)$ to be of density

$$\frac{\sqrt{\beta}}{\pi^2|y_k|}\frac{1}{1+\beta|y_k|^2}, \qquad y_k \in \mathbb{C}, \tag{11}$$

with $\beta = 1/(\tilde{\beta}|y_{k-\ell_0}|^2)$ and[6]

$$\tilde{\beta} = \min\left\{\rho^{\ell_0-1}\frac{\alpha_{\ell_0}}{\max_{0\leq\ell'\leq\ell_0}\alpha_{\ell'}}, \alpha_{\ell_0}, \rho^{\ell_0}\right\}. \tag{12}$$

With this choice

$$0 < \tilde{\beta} < 1 \qquad \text{and} \qquad \tilde{\beta}\alpha_\ell \leq \alpha_{\ell+\ell_0}, \quad \ell\in\mathbb{Z}_0^+. \tag{13}$$

Using (11) in (10), and averaging over Y_1^{k-1}, we obtain

$$\begin{aligned} I\big(X_1^n; Y_k\big|Y_1^{k-1}\big) &\leq \frac{1}{2}\mathsf{E}\left[\log|Y_k|^2\right] + \frac{1}{2}\mathsf{E}\left[\log\left(\tilde{\beta}|Y_{k-\ell_0}|^2\right)\right]\\ &\quad + \mathsf{E}\left[\log\left(\tilde{\beta}|Y_{k-\ell_0}|^2 + |Y_k|^2\right)\right] - h\big(Y_k\big|X_1^n, Y_1^{k-1}\big)\\ &\quad - \mathsf{E}\left[\log|Y_{k-\ell_0}|^2\right] + \log\frac{\pi^2}{\tilde{\beta}}. \end{aligned} \tag{14}$$

We bound the terms in (14) separately. We begin with

$$\begin{aligned} \mathsf{E}\left[\log|Y_k|^2\right] &= \mathsf{E}\left[\mathsf{E}\left[\log|Y_k|^2 \;\middle|\; X_1^k\right]\right]\\ &\leq \mathsf{E}\left[\log\left(\mathsf{E}\left[|Y_k|^2 \;\middle|\; X_1^k\right]\right)\right]\\ &= \mathsf{E}\left[\log\left(\sigma^2 + \sum_{\ell=1}^{k}\alpha_{k-\ell}|X_\ell|^2\right)\right], \end{aligned} \tag{15}$$

where the inequality follows from Jensen's inequality. Likewise, we use Jensen's inequality and (13) to upper bound

$$\begin{aligned} \mathsf{E}\left[\log\left(\tilde{\beta}|Y_{k-\ell_0}|^2\right)\right] &\leq \mathsf{E}\left[\log\left(\tilde{\beta}\sigma^2 + \tilde{\beta}\sum_{\ell=1}^{k-\ell_0}\alpha_{k-\ell_0-\ell}|X_\ell|^2\right)\right]\\ &\leq \mathsf{E}\left[\log\left(\sigma^2 + \sum_{\ell=1}^{k-\ell_0}\alpha_{k-\ell}|X_\ell|^2\right)\right] \end{aligned} \tag{16}$$

[2] See [12] for conditions that guarantee that $C(\mathrm{SNR})$ is achievable.

[3] By a stationary channel we mean a channel where for any stationary input $\{X_k\}$ the pair $\{(X_k, Y_k)\}$ is jointly stationary.

[4] Throughout this paper, $\log(\cdot)$ denotes the natural logarithm function.

[5] For channels with finite input and output alphabets this inequality follows by Topsøe's identity [15]; see also [16, Thm. 3.4].

[6] When $y_{k-\ell_0} = 0$, then the density of the Cauchy distribution (11) is undefined. However, this event is of zero probability and has therefore no impact on the mutual information $I\big(X_1^n; Y_k\big|Y_1^{k-1}\big)$.

and

$$\mathsf{E}\left[\log\left(\tilde{\beta}|Y_{k-\ell_0}|^2+|Y_k|^2\right)\right]$$
$$\leq \mathsf{E}\left[\log\left(2\sigma^2+2\sum_{\ell=1}^{k-\ell_0}\alpha_{k-\ell}|X_\ell|^2+\sum_{\ell=k-\ell_0+1}^{k}\alpha_{k-\ell}|X_\ell|^2\right)\right]$$
$$\leq \log 2+\mathsf{E}\left[\log\left(\sigma^2+\sum_{\ell=1}^{k}\alpha_{k-\ell}|X_\ell|^2\right)\right], \tag{17}$$

where the second inequality follows because $\sum_{\ell=k-\ell_0+1}^{k}\alpha_{k-\ell}|X_\ell|^2 \leq 2\sum_{\ell=k-\ell_0+1}^{k}\alpha_{k-\ell}|X_\ell|^2$.

Next, we derive a lower bound on $h\big(Y_k\big|X_1^n,Y_1^{k-1}\big)$. Let $\mathbf{H}_{k'}=\big(H_{k'}^{(0)},H_{k'}^{(1)},\ldots,H_{k'}^{(k-1)}\big)$, $k'=1,2,\ldots,k-1$. We have

$$h\big(Y_k\big|X_1^n,Y_1^{k-1}\big)\geq h\big(Y_k\big|X_1^n,Y_1^{k-1},\mathbf{H}_1^{k-1}\big)$$
$$= h\big(Y_k\big|X_1^n,\mathbf{H}_1^{k-1}\big), \tag{18}$$

where the inequality follows because conditioning cannot increase entropy, and where the equality follows because, conditional on $\big(X_1^n,\mathbf{H}_1^{k-1}\big)$, Y_k is independent of Y_1^{k-1}. Let $\mathcal{S}_k$ be defined as

$$\mathcal{S}_k \triangleq \{\ell=1,2,\ldots,k:\min\{|x_\ell|^2,\alpha_{k-\ell}\}>0\}. \tag{19}$$

Using the entropy power inequality [13, Thm. 16.6.3], and using that the processes

$$\big\{H_k^{(0)},k\in\mathbb{Z}^+\big\},\big\{H_k^{(1)},k\in\mathbb{Z}^+\big\},\ldots$$

are independent and jointly independent of X_1^n, it can be shown that for any given $X_1^n=x_1^n$

$$h\left(\sum_{\ell=1}^{k}H_k^{(k-\ell)}X_\ell+Z_k\,\middle|\,X_1^n=x_1^n,\mathbf{H}_1^{k-1}\right)$$
$$\geq \log\left(\sum_{\ell\in\mathcal{S}_k}e^{h\left(H_k^{(k-\ell)}X_\ell\,\middle|\,X_\ell=x_\ell,\{H_{k'}^{(k-\ell)}\}_{k'=1}^{k-1}\right)}+e^{h(Z_k)}\right). \tag{20}$$

We lower bound the differential entropies on the RHS of (20) as follows. The differential entropies in the sum are lower bounded by

$$h\left(H_k^{(k-\ell)}X_\ell\,\middle|\,X_\ell=x_\ell,\{H_{k'}^{(k-\ell)}\}_{k'=1}^{k-1}\right)$$
$$=\log\left(\alpha_{k-\ell}|x_\ell|^2\right)+h\Big(H_k^{(k-\ell)}\Big|\big\{H_{k'}^{(k-\ell)}\big\}_{k'=1}^{k-1}\Big)-\log\alpha_{k-\ell}$$
$$\geq \log\left(\alpha_{k-\ell}|x_\ell|^2\right)+\inf_{\ell\in\mathcal{L}}\left(h_\ell-\log\alpha_\ell\right),\qquad \ell\in\mathcal{S}_k, \tag{21}$$

where the equality follows from the behavior of differential entropy under scaling; and where the inequality follows by the stationarity of the process $\big\{H_k^{(k-\ell)},k\in\mathbb{Z}^+\big\}$ which implies that the differential entropy $h\big(H_k^{(k-\ell)}\big|\big\{H_{k'}^{(k-\ell)}\big\}_{k'=1}^{k-1}\big)$ cannot be smaller than the differential entropy rate $h_{k-\ell}$ [13, Thms. 4.2.1 & 4.2.2], and by lower bounding $(h_{k-\ell}-\log\alpha_{k-\ell})$ by $\inf_{\ell\in\mathcal{L}}(h_\ell-\log\alpha_\ell)$ (which holds for each $\ell\in\mathcal{S}_k$ because $\mathcal{S}_k\subseteq\mathcal{L}$). The last differential entropy on the RHS of (20) is lower bounded by

$$h(Z_k)\geq\inf_{\ell\in\mathcal{L}}\left(h_\ell-\log\alpha_\ell\right)+\log\sigma^2, \tag{22}$$

which follows by noting that

$$\inf_{\ell\in\mathcal{L}}\left(h_\ell-\log\alpha_\ell\right)\leq\log(\pi e). \tag{23}$$

Applying (21) & (22) to (20), and averaging over X_1^n, yields then

$$h\big(Y_k\big|X_1^n,Y_1^{k-1}\big)\geq\mathsf{E}\left[\log\left(\sigma^2+\sum_{\ell=1}^{k}\alpha_{k-\ell}|X_\ell|^2\right)\right]$$
$$+\inf_{\ell\in\mathcal{L}}\left(h_\ell-\log\alpha_\ell\right). \tag{24}$$

We continue with the analysis of (14) by lower bounding $\mathsf{E}\big[\log|Y_{k-\ell_0}|^2\big]$. To this end, we write the expectation as

$$\mathsf{E}\left[\mathsf{E}\left[\log\left|\sum_{\ell=1}^{k-\ell_0}H_{k-\ell_0}^{(k-\ell_0-\ell)}X_\ell+Z_{k-\ell_0}\right|^2\,\middle|\,X_1^{k-\ell_0}\right]\right]$$

and lower bound the conditional expectation by

$$\mathsf{E}\left[\log\left|\sum_{\ell=1}^{k-\ell_0}H_{k-\ell_0}^{(k-\ell_0-\ell)}X_\ell+Z_{k-\ell_0}\right|^2\,\middle|\,X_1^{k-\ell_0}=x_1^{k-\ell_0}\right]$$
$$=\log\left(\sigma^2+\sum_{\ell=1}^{k-\ell_0}\alpha_{k-\ell_0-\ell}|x_\ell|^2\right)$$
$$-2\cdot\mathsf{E}\left[\log\left|\frac{\sum_{\ell=1}^{k-\ell_0}H_{k-\ell_0}^{(k-\ell_0-\ell)}x_\ell+Z_{k-\ell_0}}{\sqrt{\sigma^2+\sum_{\ell=1}^{k-\ell_0}\alpha_{k-\ell_0-\ell}|x_\ell|^2}}\right|^{-1}\right]$$
$$\geq\log\left(\sigma^2+\sum_{\ell=1}^{k-\ell_0}\alpha_{k-\ell_0-\ell}|x_\ell|^2\right)+\log\delta^2-2\epsilon(\delta,\eta)$$
$$-\frac{2}{\eta}h^-\left(\frac{\sum_{\ell=1}^{k-\ell_0}H_{k-\ell_0}^{(k-\ell_0-\ell)}x_\ell+Z_{k-\ell_0}}{\sqrt{\sigma^2+\sum_{\ell=1}^{k-\ell_0}\alpha_{k-\ell_0-\ell}|x_\ell|^2}}\right) \tag{25}$$

for some $0<\delta\leq 1$ and $0<\eta<1$, where

$$h^-(X)\triangleq\int_{\{x\in\mathbb{C}:f_X(x)>1\}}f_X(x)\log f_X(x)\,\mathrm{d}x, \tag{26}$$

and where $\epsilon(\delta,\eta)>0$ tends to zero as $\delta\downarrow 0$. (We write x_ℓ in lower case to indicate that expectation and entropy are conditional on $X_1^{k-\ell_0}=x_1^{k-\ell_0}$.) Here, the inequality follows by writing the expectation in the form $\mathsf{E}\big[\log|A|^{-1}\cdot I\{|A|>\delta\}\big]+\mathsf{E}\big[\log|A|^{-1}\cdot I\{|A|\leq\delta\}\big]$ (where $I\{\cdot\}$ denotes the indicator function), and by upper bounding then the first expectation by $-\log\delta$ and the second expectation using [5, Lemma 6.7]. We continue by upper bounding

$$h^-\left(\frac{\sum_{\ell=1}^{k-\ell_0}H_{k-\ell_0}^{(k-\ell_0-\ell)}x_\ell+Z_{k-\ell_0}}{\sqrt{\sigma^2+\sum_{\ell=1}^{k-\ell_0}\alpha_{k-\ell_0-\ell}|x_\ell|^2}}\right)$$
$$=h^+\left(\frac{\sum_{\ell=1}^{k-\ell_0}H_{k-\ell_0}^{(k-\ell_0-\ell)}x_\ell+Z_{k-\ell_0}}{\sqrt{\sigma^2+\sum_{\ell=1}^{k-\ell_0}\alpha_{k-\ell_0-\ell}|x_\ell|^2}}\right)$$

$$- h\left(\frac{\sum_{\ell=1}^{k-\ell_0} H_{k-\ell_0}^{(k-\ell_0-\ell)} x_\ell + Z_{k-\ell_0}}{\sqrt{\sigma^2 + \sum_{\ell=1}^{k-\ell_0} \alpha_{k-\ell_0-\ell}|x_\ell|^2}}\right)$$
$$\leq \frac{2}{e} + \log(\pi e) + \log\left(\sigma^2 + \sum_{\ell=1}^{k-\ell_0} \alpha_{k-\ell_0-\ell}|x_\ell|^2\right)$$
$$- h\left(\sum_{\ell=1}^{k-\ell_0} H_{k-\ell_0}^{(k-\ell_0-\ell)} x_\ell + Z_{k-\ell_0}\right), \tag{27}$$

where $h^+(X)$ is defined as $h^+(X) \triangleq h(X) + h^-(X)$. Here, we applied [5, Lemma 6.4] to upper bound

$$h^+\left(\frac{\sum_{\ell=1}^{k-\ell_0} H_{k-\ell_0}^{(k-\ell_0-\ell)} x_\ell + Z_{k-\ell_0}}{\sqrt{\sigma^2 + \sum_{\ell=1}^{k-\ell_0} \alpha_{k-\ell_0-\ell}|x_\ell|^2}}\right) \leq \frac{2}{e} + \log(\pi e). \tag{28}$$

Averaging (27) over $X_1^{k-\ell_0}$ yields

$$h^-\left(\frac{\sum_{\ell=1}^{k-\ell_0} H_{k-\ell_0}^{(k-\ell_0-\ell)} X_\ell + Z_{k-\ell_0}}{\sqrt{\sigma^2 + \sum_{\ell=1}^{k-\ell_0} \alpha_{k-\ell_0-\ell}|X_\ell|^2}}\middle| X_1^{k-\ell_0}\right)$$
$$\leq \frac{2}{e} + \log(\pi e) + \mathsf{E}\left[\log\left(\sigma^2 + \sum_{\ell=1}^{k-\ell_0} \alpha_{k-\ell_0-\ell}|X_\ell|^2\right)\right]$$
$$- h\left(\sum_{\ell=1}^{k-\ell_0} H_{k-\ell_0}^{(k-\ell_0-\ell)} X_\ell + Z_{k-\ell_0}\middle| X_1^{k-\ell_0}\right)$$
$$\leq \frac{2}{e} + \log(\pi e) - \inf_{\ell\in\mathcal{L}} (h_\ell - \log\alpha_\ell), \tag{29}$$

where the second inequality follows by conditioning the differential entropy additionally on $Y_1^{k-\ell_0-1}$, and by using then lower bound (24). A lower bound on $\mathsf{E}\left[\log|Y_{k-\ell_0}|^2\right]$ follows now by averaging (25) over $X_1^{k-\ell_0}$, and by applying (29)

$$\mathsf{E}\left[\log|Y_{k-\ell_0}|^2\right] \geq \mathsf{E}\left[\log\left(\sigma^2 + \sum_{\ell=1}^{k-\ell_0} \alpha_{k-\ell_0-\ell}|X_\ell|^2\right)\right]$$
$$+ \log\delta^2 - 2\epsilon(\delta,\eta) - \frac{2}{\eta}\left(\frac{2}{e} + \log(\pi e)\right)$$
$$+ \inf_{\ell\in\mathcal{L}} \frac{2}{\eta}(h_\ell - \log\alpha_\ell). \tag{30}$$

Returning to the analysis of (14), we obtain from (30), (24), (17), (16), and (15)

$$I\left(X_1^n; Y_k \middle| Y_1^{k-1}\right)$$
$$\leq \frac{1}{2}\mathsf{E}\left[\log\left(\sigma^2 + \sum_{\ell=1}^{k} \alpha_{k-\ell}|X_\ell|^2\right)\right]$$
$$+ \frac{1}{2}\mathsf{E}\left[\log\left(\sigma^2 + \sum_{\ell=1}^{k-\ell_0} \alpha_{k-\ell}|X_\ell|^2\right)\right]$$
$$+ \log 2 + \mathsf{E}\left[\log\left(\sigma^2 + \sum_{\ell=1}^{k} \alpha_{k-\ell}|X_\ell|^2\right)\right]$$
$$- \mathsf{E}\left[\log\left(\sigma^2 + \sum_{\ell=1}^{k} \alpha_{k-\ell}|X_\ell|^2\right)\right] - \inf_{\ell\in\mathcal{L}}(h_\ell - \log\alpha_\ell)$$
$$- \mathsf{E}\left[\log\left(\sigma^2 + \sum_{\ell=1}^{k-\ell_0} \alpha_{k-\ell_0-\ell}|X_\ell|^2\right)\right]$$
$$- \log\delta^2 + 2\epsilon(\delta,\eta) + \frac{2}{\eta}\left(\frac{2}{e} + \log(\pi e)\right)$$
$$- \inf_{\ell\in\mathcal{L}} \frac{2}{\eta}(h_\ell - \log\alpha_\ell) + \log\frac{\pi^2}{\tilde{\beta}}$$
$$\leq \mathsf{K} + \mathsf{E}\left[\log\left(\sigma^2 + \sum_{\ell=1}^{k} \alpha_{k-\ell}|X_\ell|^2\right)\right]$$
$$- \mathsf{E}\left[\log\left(\sigma^2 + \sum_{\ell=1}^{k-\ell_0} \alpha_{k-\ell_0-\ell}|X_\ell|^2\right)\right], \tag{31}$$

with

$$\mathsf{K} \triangleq -\inf_{\ell\in\mathcal{L}}\left(1 + \frac{2}{\eta}\right)(h_\ell - \log\alpha_\ell) + \log\frac{2\pi^2}{\tilde{\beta}\delta^2}$$
$$+ 2\epsilon(\delta,\eta) + \frac{2}{\eta}\left(\frac{2}{e} + \log(\pi e)\right). \tag{32}$$

The second inequality in (31) follows because $\sum_{\ell=1}^{k-\ell_0} \alpha_{k-\ell}|X_\ell|^2 \leq \sum_{\ell=1}^{k} \alpha_{k-\ell}|X_\ell|^2$.

In order to show that the capacity is bounded in the SNR, we apply (31) and (9) to (8) and use then that for any sequences $\{a_k\}$ and $\{b_k\}$

$$\sum_{k=\ell_0+1}^{n} (a_k - b_k) = \sum_{k=n-\ell_0+1}^{n} (a_k - b_{k-n+2\ell_0})$$
$$+ \sum_{k=\ell_0+1}^{n-\ell_0} (a_k - b_{k+\ell_0}). \tag{33}$$

Defining

$$a_k \triangleq \mathsf{E}\left[\log\left(\sigma^2 + \sum_{\ell=1}^{k} \alpha_{k-\ell}|X_\ell|^2\right)\right] \tag{34}$$

and

$$b_k \triangleq \mathsf{E}\left[\log\left(\sigma^2 + \sum_{\ell=1}^{k-\ell_0} \alpha_{k-\ell_0-\ell}|X_\ell|^2\right)\right] \tag{35}$$

we have for the first sum on the RHS of (33)

$$\sum_{k=n-\ell_0+1}^{n} (a_k - b_{k-n+2\ell_0})$$
$$= \sum_{k=n-\ell_0+1}^{n} \mathsf{E}\left[\log\left(\frac{\sigma^2 + \sum_{\ell=1}^{k} \alpha_{k-\ell}|X_\ell|^2}{\sigma^2 + \sum_{\ell=1}^{k-n+\ell_0} \alpha_{k-n+\ell_0-\ell}|X_\ell|^2}\right)\right]$$
$$\leq \ell_0 \log\left(1 + \sup_{\ell\in\mathbb{Z}_0^+} \alpha_\ell \cdot n \cdot \mathsf{SNR}\right), \tag{36}$$

which follows by lower bounding the denominator by σ^2, and by using then Jensen's inequality along with the last inequality in (9). For the second sum on the RHS of (33) we have

$$\sum_{k=\ell_0+1}^{n-\ell_0} (a_k - b_{k+\ell_0}) = \sum_{k=\ell_0+1}^{n-\ell_0} \mathsf{E}\left[\log\left(\frac{\sigma^2 + \sum_{\ell=1}^{k}\alpha_{k-\ell}|X_\ell|^2}{\sigma^2 + \sum_{\ell=1}^{k}\alpha_{k-\ell}|X_\ell|^2}\right)\right] = 0. \quad (37)$$

Thus, applying (31)–(37) and (9) to (8), we obtain

$$\frac{1}{n} I(X_1^n; Y_1^n) \leq \frac{\ell_0}{n}\log\left(\sigma^2 + \sup_{\ell\in\mathbb{Z}_0^+}\alpha_\ell \cdot n \cdot \mathrm{SNR}\right) + \frac{\ell_0}{n}\log\left(\sigma^2 + \sup_{\ell\in\mathbb{Z}_0^+}\alpha_\ell \cdot n\cdot \mathrm{SNR}\right) + \frac{n-2\ell_0}{n}\mathrm{K}, \quad (38)$$

which tends to $\mathrm{K} < \infty$ as n tends to infinity. This proves Theorem 1.

III. Multipath Channels of Unbounded Capacity

We have seen in Theorem 1 that if the variances of the path gains $\{\alpha_\ell\}$ do not decay faster than geometrically, then capacity is bounded in the SNR. In this section, we demonstrate that this need not be the case when the variances of the path gains decay faster than geometrically. The following theorem presents a sufficient condition for the capacity $C(\mathrm{SNR})$ to be unbounded in the SNR.

Theorem 2: Consider the above channel model. Then

$$\left(\lim_{\ell\to\infty}\frac{1}{\ell}\log\log\frac{1}{\alpha_\ell} = \infty\right) \Longrightarrow \left(\sup_{\mathrm{SNR}>0} C(\mathrm{SNR}) = \infty\right). \quad (39)$$

Proof: Omitted. ∎

Note: We do not claim that $C(\mathrm{SNR})$ is achievable. However, it can be shown that when, for example, the processes $\{H_k^{(\ell)}, k \in \mathbb{Z}^+\}$, $\ell \in \mathbb{Z}_0^+$ are IID Gaussian, then the maximum achievable rate is unbounded in the SNR, i.e., any rate is achievable for sufficiently large SNR.

Certainly, the condition on the LHS of (39) is satisfied when the channel has finite memory in the sense that for some finite $L \in \mathbb{Z}_0^+$

$$\alpha_\ell = 0, \qquad \ell = L+1, L+2, \ldots.$$

In this case, (1) becomes

$$Y_k = \begin{cases} \sum_{\ell=0}^{k-1} H_k^{(\ell)} x_{k-\ell} + Z_k, & k = 1,2,\ldots,L \\ \sum_{\ell=0}^{L} H_k^{(\ell)} x_{k-\ell} + Z_k, & k = L+1, L+2, \ldots. \end{cases} \quad (40)$$

This channel (40) was studied for general (but finite) L in [17] where it was shown that its capacity satisfies

$$\lim_{\mathrm{SNR}\to\infty}\frac{C(\mathrm{SNR})}{\log\log\mathrm{SNR}} = 1. \quad (41)$$

Thus, for finite L, the capacity pre-loglog (41) is not affected by the multipath behavior. This is perhaps surprising as Theorem 1 implies that if $L = \infty$, and if the variances of the path gains do not decay faster than geometrically, then the pre-loglog is zero.

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