MULTIPATH CHANNELS OF UNBOUNDED CAPACITY

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ABSTRACT

The capacity of discrete-time, noncoherent, multipath fading channels is considered. It is shown that if the variances of the path gains decay faster than exponentially, then capacity is unbounded in the transmit power.

Index Terms— Channel capacity, information rates, multipath channels, fading channels, noncoherent.

1. INTRODUCTION

This paper studies the capacity of multipath (frequency-selective) fading channels. A noncoherent channel model is considered where neither transmitter nor receiver are cognizant of the fading's realization, but both are aware of its statistic. Our focus is on the high signal-to-noise ratio (SNR) regime.

For the special case of noncoherent *frequency-flat* fading channels (where we have only *one* path), it was shown by Lapidoth & Moser [1] that if the fading process is of finite entropy rate, then at high SNR capacity grows double-logarithmically with the SNR. This is in stark contrast to the logarithmic growth of the capacity of coherent fading channels (where the realization of the fading is known to the receiver) [2]. Thus, communicating over noncoherent flat-fading channels at high SNR is power inefficient.

Recently, it has been demonstrated that communicating over noncoherent *multipath* fading channels at high SNR is not merely power inefficient, but may be even worse: if the delay spread is large in the sense that the variances of the path gains decay *exponentially or slower*, then capacity is *bounded* in the SNR; see [3, Thm. 1]. For such channels, capacity does not tend to infinity as the SNR tends to infinity.

In contrast, if the variances of the path gains decay *faster than double-exponentially*, then capacity is *unbounded* in the SNR; see [3, Thm. 2]. This condition is certainly satisfied if the number of paths is finite, i.e., if the channel output is only influenced by the present and by the L previous channel inputs. (Here only the variances of the first $(\mathsf{L}+1)$ path gains are positive, while the other variances are zero.) It was shown in [4] that in this case capacity is not only unbounded in the SNR, but its growth with the SNR is also independent of the number of paths and equals the growth of the capacity of noncoherent frequency-flat fading channels, i.e.,

$$\lim_{\text{SNR}\to\infty} \frac{C(\text{SNR})}{\log\log\text{SNR}} = 1.$$

Thus, for finite L, the capacity pre-loglog is unaffected by the number of paths L.

The above results demonstrate that whether the capacity of a multipath channel is unbounded in the SNR depends critically on the decay rate of the variances of the path gains. However, [3, Thm. 1] only accounts for decay rates that are exponentially or slower, whereas [3, Thm. 2] only regards decay rates that are faster than double-exponentially. Thus, [3, Thm. 1] & [3, Thm. 2] fail to characterize the capacity of channels for which the variances of the path gains decay faster than exponentially but slower than double-exponentially. In this paper, we bridge this gap by showing that if the variances of the path gains decay faster than exponentially, then capacity is unbounded in the SNR.

1.1. Channel Model

Let $\mathbb{C}$ and $\mathbb{N}$ denote the set of complex numbers and the set of positive integers, respectively. We consider a discrete-time multipath fading channel whose channel output $Y_k \in \mathbb{C}$ at time $k \in \mathbb{N}$ corresponding to the time-1 through time-k channel inputs $x_1, \ldots, x_k \in \mathbb{C}$ is given by

$$Y_k = \sum_{\ell=0}^{k-1} H_k^{(\ell)} x_{k-\ell} + Z_k, \quad k \in \mathbb{N}. \tag{1}$$

Here $\{Z_k\}$ models additive noise, and $H_k^{(\ell)}$ denotes the time-k gain of the ℓ-th path. We assume that $\{Z_k\}$ is a sequence of independent and identically distributed (IID), zero-mean, variance-σ^2, circularly-symmetric, complex Gaussian random variables. For each path $\ell \in \mathbb{N}_0$ (where $\mathbb{N}_0$ denotes the set of nonnegative integers), we assume that $\{H_k^{(\ell)},\ k \in \mathbb{N}\}$ is a zero-mean, complex stationary process. We denote its variance and its differential entropy rate by

$$\alpha_\ell \triangleq \mathsf{E}\left[\left|H_k^{(\ell)}\right|^2\right], \qquad \ell \in \mathbb{N}_0 \tag{2}$$

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and

$$h_\ell \triangleq \lim_{n\to\infty} \frac{1}{n} h\big(H_1^{(\ell)}, \ldots, H_n^{(\ell)}\big), \quad \ell \in \mathbb{N}_0. \tag{3}$$

Without loss of generality we assume that $\alpha_0 > 0$. We further assume that

$$\sum_{\ell=0}^{\infty} \alpha_\ell \triangleq \alpha < \infty \tag{4}$$

and

$$\inf_{\ell \in \mathcal{L}} h_\ell > -\infty, \tag{5}$$

where the set $\mathcal{L}$ is defined as $\mathcal{L} \triangleq \{\ell \in \mathbb{N}_0 : \alpha_\ell > 0\}$. We finally assume that the processes

$$\big\{H_k^{(0)},\ k \in \mathbb{N}\big\}, \big\{H_k^{(1)},\ k \in \mathbb{N}\big\}, \ldots$$

are independent ("uncorrelated scattering"); that they are jointly independent of $\{Z_k\}$; and that the joint law of

$$\Big(\{Z_k\}, \big\{H_k^{(0)},\ k \in \mathbb{N}\big\}, \big\{H_k^{(1)},\ k \in \mathbb{N}\big\}, \ldots\Big)$$

does not depend on the input sequence $\{x_k\}$. We consider a noncoherent channel model where neither transmitter nor receiver is cognizant of the realization of $\big\{H_k^{(\ell)},\ k \in \mathbb{N}\big\}$, $\ell \in \mathbb{N}_0$, but both are aware of their law. We do not assume that the path gains are Gaussian.

1.2. Channel Capacity

Let A_m^n denote the sequence $A_m, \ldots, A_n$. We define the capacity as

$$C(\text{SNR}) \triangleq \varliminf_{n\to\infty} \frac{1}{n} \sup I\big(X_1^n; Y_1^n\big), \tag{6}$$

where the supremum is over all joint distributions on $X_1, \ldots, X_n$ satisfying the power constraint

$$\frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\big[|X_k|^2\big] \le \mathsf{P}, \tag{7}$$

and where SNR is defined as

$$\text{SNR} \triangleq \frac{\mathsf{P}}{\sigma^2}. \tag{8}$$

By Fano's inequality, no rate above $C(\text{SNR})$ is achievable. (See [5] for a definition of an achievable rate.) We do not claim that there is a coding theorem associated with (6), i.e., that $C(\text{SNR})$ is achievable. A coding theorem will hold, for example, if there are only $(\mathsf{L}+1)$ paths (for some $\mathsf{L} < \infty$), and if the processes corresponding to these paths

$$\big\{H_k^{(0)},\ k \in \mathbb{N}\big\}, \ldots, \big\{H_k^{(\mathsf{L})},\ k \in \mathbb{N}\big\}$$

are jointly ergodic, see [6].

In [3] a necessary and a sufficient condition for $C(\text{SNR})$ to be bounded in SNR was derived:

Theorem 1. *Consider the above channel model. Then*

$$\left(\varliminf_{\ell\to\infty} \frac{\alpha_{\ell+1}}{\alpha_\ell} > 0\right) \Longrightarrow \left(\sup_{\text{SNR}>0} C(\text{SNR}) < \infty\right) \tag{9}$$

and

$$\left(\lim_{\ell\to\infty} \frac{1}{\ell} \log\log \frac{1}{\alpha_\ell} = \infty\right) \Longrightarrow \left(\sup_{\text{SNR}>0} C(\text{SNR}) = \infty\right), \tag{10}$$

where we define $a/0 \triangleq \infty$ for every $a > 0$ and $0/0 \triangleq 0$.

Proof. For the first condition (9) see [3, Thm. 1], and for the second condition (10) see [3, Thm. 2]. □

For example, when $\alpha_\ell = e^{-\ell}$, then capacity is bounded, and when $\alpha_\ell = \exp\big(-\exp(\ell^\kappa)\big)$ for some $\kappa > 1$, then capacity is unbounded. Roughly speaking, we can say that when $\{\alpha_\ell\}$ decays exponentially or slower, then $C(\text{SNR})$ is bounded in SNR, and when $\{\alpha_\ell\}$ decays faster than double-exponentially, then $C(\text{SNR})$ is unbounded in SNR.

1.3. Main Result

Our main result is an improved achievability result. We derive a weaker condition that satisfies to guarantee that capacity is unbounded in the SNR.

Theorem 2. *Consider the above channel model. Then*

$$\left(\lim_{\ell\to\infty} \frac{1}{\ell} \log \frac{1}{\alpha_\ell} = \infty\right) \Longrightarrow \left(\sup_{\text{SNR}>0} C(\text{SNR}) = \infty\right), \tag{11}$$

where we define $1/0 \triangleq \infty$.

Proof. See Section 2. □

By noting that

$$\left(\lim_{\ell\to\infty} \frac{\alpha_{\ell+1}}{\alpha_\ell} = 0\right) \Longrightarrow \left(\lim_{\ell\to\infty} \frac{1}{\ell} \log \frac{1}{\alpha_\ell} = \infty\right)$$

we obtain from Theorems 1 & 2 the immediate corollary:

Corollary 3. *Consider the above channel model. Then*

$$i) \left(\varliminf_{\ell\to\infty} \frac{\alpha_{\ell+1}}{\alpha_\ell} > 0\right) \Longrightarrow \left(\sup_{\text{SNR}>0} C(\text{SNR}) < \infty\right) \tag{12}$$

$$ii) \left(\lim_{\ell\to\infty} \frac{\alpha_{\ell+1}}{\alpha_\ell} = 0\right) \Longrightarrow \left(\sup_{\text{SNR}>0} C(\text{SNR}) = \infty\right), \tag{13}$$

where we define $a/0 \triangleq \infty$ for every $a > 0$ and $0/0 \triangleq 0$.

For example, when $\alpha_\ell = \exp(-\ell^\kappa)$ for some $\kappa > 1$, then capacity is unbounded.

Theorem 2 and Corollary 3 demonstrate that when $\{\alpha_\ell\}$ decays faster than exponentially, then $C(\text{SNR})$ is unbounded in SNR, thus bridging the gap between (9) and (10).

2. PROOF OF THEOREM 2

In order to prove Theorem 2, we shall derive in Section 2.1 a lower bound on capacity and then show in Section 2.2 that this bound can be made arbitrarily large, provided that

$$\lim_{\ell\to\infty}\frac{1}{\ell}\log\frac{1}{\alpha_\ell}=\infty.$$

2.1. Capacity Lower Bound

To derive a lower bound on capacity, we evaluate $\frac{1}{n}I(X_1^n;Y_1^n)$ for the following distribution on the inputs $\{X_k\}$.

Let $\mathsf{L}(\mathsf{P})$ be such that

$$\sum_{\ell=\mathsf{L}(\mathsf{P})+1}^{\infty}\alpha_\ell\cdot\mathsf{P}\leq\sigma^2. \tag{14}$$

To shorten notation, we shall write in the following L instead of $\mathsf{L}(\mathsf{P})$. Let $\tau\in\mathbb{N}$ be some positive integer that possibly depends on L, and let $\mathbf{X}_b=(X_{b(\mathsf{L}+\tau)+1},\ldots,X_{(b+1)(\mathsf{L}+\tau)})$. We choose $\{\mathbf{X}_b\}$ to be IID with

$$\mathbf{X}_b=\big(\underbrace{0,\ldots,0}_{\mathsf{L}},\tilde{X}_{b\tau+1},\ldots,\tilde{X}_{(b+1)\tau}\big),$$

where $\tilde{X}_{b\tau+1},\ldots,\tilde{X}_{(b+1)\tau}$ is a sequence of independent, zero-mean, circularly-symmetric, complex random variables with $\log|\tilde{X}_{b\tau+\nu}|^2$ being uniformly distributed over the interval $\big[\log\mathsf{P}^{(\nu-1)/\tau},\log\mathsf{P}^{\nu/\tau}\big]$, i.e., for each $\nu=1,\ldots,\tau$

$$\log|\tilde{X}_{b\tau+\nu}|^2\sim\mathcal{U}\left(\big[\log\mathsf{P}^{(\nu-1)/\tau},\log\mathsf{P}^{\nu/\tau}\big]\right).$$

(Here and throughout this proof we assume that $\mathsf{P}>1$.)

Let $\kappa\triangleq\lfloor\frac{n}{\mathsf{L}+\tau}\rfloor$ (where $\lfloor a\rfloor$ denotes the largest integer that is less than or equal to a), and let $\mathbf{Y}_b$ denote the vector $(Y_{b(L+\tau)+1},\ldots,Y_{(b+1)(\mathsf{L}+\tau)})$. By the chain rule for mutual information [5, Thm. 2.5.2] we have

$$\begin{aligned}I\big(X_1^n;Y_1^n\big)&\geq I\big(\mathbf{X}_0^{\kappa-1};\mathbf{Y}_0^{\kappa-1}\big)\\&=\sum_{b=0}^{\kappa-1}I\big(\mathbf{X}_b;\mathbf{Y}_0^{\kappa-1}\bigm|\mathbf{X}_0^{b-1}\big)\\&\geq\sum_{b=0}^{\kappa-1}I(\mathbf{X}_b;\mathbf{Y}_b),\end{aligned}\tag{15}$$

where the first inequality follows by restricting the number of observables; and where the last inequality follows by restricting the number of observables and by noting that $\{\mathbf{X}_b\}$ is IID.

We continue by lower bounding each summand on the right-hand side (RHS) of (15). We use again the chain rule and that reducing observations cannot increase mutual information to obtain

$$\begin{aligned}I(\mathbf{X}_b;\mathbf{Y}_b)&=\sum_{\nu=1}^{\tau}I\big(\tilde{X}_{b\tau+\nu};\mathbf{Y}_b\bigm|\tilde{X}_{b\tau+1}^{b\tau+\nu-1}\big)\\&\geq\sum_{\nu=1}^{\tau}I\big(\tilde{X}_{b\tau+\nu};Y_{b(\mathsf{L}+\tau)+\mathsf{L}+\nu}\bigm|\tilde{X}_{b\tau+1}^{b\tau+\nu-1}\big)\\&\geq\sum_{\nu=1}^{\tau}I\big(\tilde{X}_{b\tau+\nu};Y_{b(\mathsf{L}+\tau)+\mathsf{L}+\nu}\big),\end{aligned}\tag{16}$$

where we have additionally used in the last inequality that $\tilde{X}_{b\tau+1},\ldots,\tilde{X}_{(b+1)\tau}$ are independent.

Defining

$$\begin{aligned}W_{b\tau+\nu}\triangleq\sum_{\ell=1}^{b(\mathsf{L}+\tau)+\mathsf{L}+\nu-1}&H_{b(\mathsf{L}+\tau)+\mathsf{L}+\nu}^{(\ell)}X_{b(\mathsf{L}+\tau)+\mathsf{L}+\nu-\ell}\\&+Z_{b(\mathsf{L}+\tau)+\mathsf{L}+\nu}\end{aligned}\tag{17}$$

each summand on the RHS of (16) can be written as

$$\begin{aligned}&I\big(\tilde{X}_{b\tau+\nu};Y_{b(\mathsf{L}+\tau)+\mathsf{L}+\nu}\big)\\&\quad=I\big(\tilde{X}_{b\tau+\nu};H_{b(\mathsf{L}+\tau)+\mathsf{L}+\nu}^{(0)}\tilde{X}_{b\tau+\nu}+W_{b\tau+\nu}\big).\end{aligned}\tag{18}$$

A lower bound on (18) follows from the following lemma.

Lemma 4. *Let the random variables X, H, and W have finite second moments. Assume that both X and H are of finite differential entropy. Finally, assume that X is independent of H; that X is independent of W; and that X —∘— H —∘— W forms a Markov chain. Then*

$$\begin{aligned}I(X;HX+W)\geq h(X)&-\mathsf{E}\big[\log|X|^2\big]+\mathsf{E}\big[\log|H|^2\big]\\&-\mathsf{E}\left[\log\left(\pi e\left(\sigma_H+\frac{\sigma_W}{|X|}\right)^2\right)\right],\end{aligned}\tag{19}$$

where $\sigma_H^2\geq0$ and $\sigma_H^2>0$ denote the variances of W and H. (Note that the assumptions that X and H have finite second moments and are of finite differential entropy guarantee that $\mathsf{E}\big[\log|X|^2\big]$ and $\mathsf{E}\big[\log|H|^2\big]$ are finite, see [1, Lemma 6.7e]*.)*

Proof. See [7, Lemma 4]. □

It can be easily verified that for the channel model given in Section 1.1 and for the above coding scheme the lemma's conditions are satisfied. We therefore obtain from Lemma 4

$$\begin{aligned}&I\big(\tilde{X}_{b\tau+\nu};H_{b(\mathsf{L}+\tau)+\mathsf{L}+\nu}^{(0)}\tilde{X}_{b\tau+\nu}+W_{b\tau+\nu}\big)\\&\geq h\big(\tilde{X}_{b\tau+\nu}\big)-\mathsf{E}\Big[\log|\tilde{X}_{b\tau+\nu}|^2\Big]+\mathsf{E}\Big[\log\big|H_{b(\mathsf{L}+\tau)+\mathsf{L}+\nu}^{(0)}\big|^2\Big]\\&\quad-\mathsf{E}\left[\log\left(\pi e\left(\sqrt{\alpha_0}+\frac{\sqrt{\mathsf{E}[|W_{b\tau+\nu}|^2]}}{|\tilde{X}_{b\tau+\nu}|}\right)^2\right)\right].\end{aligned}\tag{20}$$

Using that the differential entropy of a circularly-symmetric random variable is given by (see [1, Eqs. (320) & (316)])

$$h\big(\tilde{X}_{b\tau+\nu}\big) = \mathsf{E}\Big[\log|\tilde{X}_{b\tau+\nu}|^2\Big] + h\big(\log|\tilde{X}_{b\tau+\nu}|^2\big) + \log\pi, \tag{21}$$

and evaluating $h(\log|\tilde{X}_{b\tau+\nu}|^2)$ for our choice of $\tilde{X}_{b\tau+\nu}$, yields for the first two terms on the RHS of (20)

$$h\big(\tilde{X}_{b\tau+\nu}\big) - \mathsf{E}\Big[\log|\tilde{X}_{b\tau+\nu}|^2\Big] = \log\log\mathsf{P}^{1/\tau} + \log\pi. \tag{22}$$

We next upper bound

$$\frac{\mathsf{E}\big[|W_{b\tau+\nu}|^2\big]}{|\tilde{X}_{b\tau+\nu}|^2} = \sum_{\ell=1}^{\mathsf{L}}\alpha_\ell\frac{\mathsf{E}\big[|X_{b(\mathsf{L}+\tau)+\mathsf{L}+\nu-\ell}|^2\big]}{|\tilde{X}_{b\tau+\nu}|^2} + \sum_{\ell=\mathsf{L}+1}^{b(\mathsf{L}+\tau)+\mathsf{L}+\nu-1}\alpha_\ell\frac{\mathsf{E}\big[|X_{b(\mathsf{L}+\tau)+\mathsf{L}+\nu-\ell}|^2\big]}{|\tilde{X}_{b\tau+\nu}|^2} + \frac{\sigma^2}{|\tilde{X}_{b\tau+\nu}|^2}. \tag{23}$$

To this end we note that for our choice of $\{X_k\}$ and by the assumption that $\mathsf{P} > 1$, we have

$$\mathsf{E}\big[|X_\ell|^2\big] \leq \mathsf{P}, \quad \ell\in\mathbb{N}, \tag{24}$$

$$\mathsf{E}\big[|X_{b(\mathsf{L}+\tau)+\mathsf{L}+\nu-\ell}|^2\big] \leq \mathsf{P}^{(\nu-\ell)/\tau}, \quad \ell = 1,\ldots,\mathsf{L}, \tag{25}$$

and

$$|\tilde{X}_{b\tau+\nu}|^2 \geq \mathsf{P}^{(\nu-1)/\tau} \geq 1, \tag{26}$$

from which we obtain

$$\frac{\mathsf{E}\big[|X_{b(\mathsf{L}+\tau)+\mathsf{L}+\nu-\ell}|^2\big]}{|\tilde{X}_{b\tau+\nu}|^2} \leq \frac{\mathsf{P}^{(\nu-\ell)/\tau}}{\mathsf{P}^{(\nu-1)/\tau}} \leq 1, \quad \ell = 1,\ldots,\mathsf{L} \tag{27}$$

and

$$\frac{\mathsf{E}\big[|X_{b(\mathsf{L}+\tau)+\mathsf{L}+\nu-\ell}|^2\big]}{|\tilde{X}_{b\tau+\nu}|^2} \leq \mathsf{P}, \quad \mathsf{L} < \ell < b(\mathsf{L}+\tau)+\mathsf{L}+\nu. \tag{28}$$

Applying (26)–(28) to (23) yields

$$\begin{aligned}\frac{\mathsf{E}\big[|W_{b\tau+\nu}|^2\big]}{|\tilde{X}_{b\tau+\nu}|^2} &\leq \sum_{\ell=1}^{\mathsf{L}}\alpha_\ell + \sum_{\ell=\mathsf{L}+1}^{b(\mathsf{L}+\tau)+\mathsf{L}+\nu-1}\alpha_\ell\cdot\mathsf{P} + \sigma^2\\ &\leq \alpha + \sum_{\ell=\mathsf{L}+1}^{\infty}\alpha_\ell\cdot\mathsf{P} + \sigma^2\\ &\leq \alpha + 2\sigma^2,\end{aligned} \tag{29}$$

with α being defined in (4). Here the second inequality follows because α_ℓ, $\ell\in\mathbb{N}_0$ and P are nonnegative, and the last inequality follows from (14).

By combining (20) with (22) & (29), and by noting that by the stationarity of $\big\{H_k^{(0)},\ k\in\mathbb{N}\big\}$

$$\mathsf{E}\Big[\log\big|H^{(0)}_{b(\mathsf{L}+\tau)+\mathsf{L}+\nu}\big|^2\Big] = \mathsf{E}\Big[\log\big|H_1^{(0)}\big|^2\Big],$$

we obtain the lower bound

$$\begin{aligned}&I\big(\tilde{X}_{b\tau+\nu}; H^{(0)}_{b(\mathsf{L}+\tau)+\mathsf{L}+\nu}\tilde{X}_{b\tau+\nu} + W_{b\tau+\nu}\big)\\ &\quad\geq \log\log\mathsf{P}^{1/\tau} + \mathsf{E}\Big[\log\big|H_1^{(0)}\big|^2\Big] - 1\\ &\qquad - 2\log\big(\sqrt{\alpha_0} + \sqrt{\alpha+2\sigma^2}\big).\end{aligned} \tag{30}$$

Note that the RHS of (30) neither depends on ν nor on b. We therefore have from (30), (16), and (15)

$$I\big(X_1^n; Y_1^n\big) \geq \kappa\tau\log\log\mathsf{P}^{1/\tau} + \kappa\tau\Upsilon, \tag{31}$$

where we define Υ as

$$\Upsilon \triangleq \mathsf{E}\Big[\log\big|H_1^{(0)}\big|^2\Big] - 1 - 2\log\big(\sqrt{\alpha_0} + \sqrt{\alpha+2\sigma^2}\big). \tag{32}$$

Dividing the RHS of (31) by n, and computing the limit as n tends to infinity, yields the lower bound on capacity

$$C(\mathrm{SNR}) \geq \frac{\tau}{\mathsf{L}+\tau}\log\log\mathsf{P}^{1/\tau} + \frac{\tau}{\mathsf{L}+\tau}\Upsilon, \quad \mathsf{P} > 1, \tag{33}$$

where we have used that $\lim_{n\to\infty}\kappa/n = 1/(\mathsf{L}+\tau)$.

2.2. Unbounded Capacity

We next show that

$$\lim_{\ell\to\infty}\frac{1}{\ell}\log\frac{1}{\alpha_\ell} = \infty \tag{34}$$

implies that the RHS of (33) can be made arbitrarily large. To this end we note that by (34) we can find for every $0 < \varrho < 1$ an $\ell_0\in\mathbb{N}$ such that

$$\alpha_\ell < \varrho^\ell, \quad \ell > \ell_0. \tag{35}$$

We therefore have

$$\sum_{\ell=\ell'+1}^{\infty}\alpha_\ell < \sum_{\ell=\ell'+1}^{\infty}\varrho^\ell = \varrho^{\ell'}\frac{\varrho}{1-\varrho}, \quad \ell' \geq \ell_0. \tag{36}$$

We choose L so that it satisfies

$$\varrho^{\mathsf{L}}\frac{\varrho}{1-\varrho}\mathsf{P} \leq \sigma^2, \tag{37}$$

i.e., we choose

$$\mathsf{L} = \left\lceil\frac{\log\Big(\mathrm{SNR}\frac{\varrho}{1-\varrho}\Big)}{\log\frac{1}{\varrho}}\right\rceil \tag{38}$$

(where $\lceil a\rceil$ denotes the smallest integer that is greater than or equal to a). We shall argue next that this choice also satisfies (14). Indeed, we have by (38) that L tends to infinity as $\mathrm{SNR}\to\infty$, which implies that, for sufficiently large SNR, L is greater than ℓ_0. It follows then from (36) and (37) that

$$\sum_{\ell=\mathsf{L}+1}^{\infty}\alpha_\ell\cdot\mathsf{P} < \varrho^{\mathsf{L}}\frac{\varrho}{1-\varrho}\mathsf{P} \leq \sigma^2. \tag{39}$$

We continue by evaluating the RHS of (33) for our choice of L (38) and for $\tau = \mathsf{L}$

$$\begin{aligned} C(\mathrm{SNR}) &\geq \frac{\tau}{\mathsf{L}+\tau} \log\log \mathsf{P}^{1/\tau} + \frac{\tau}{\mathsf{L}+\tau}\Upsilon \\ &= \frac{1}{2}\log\left(\frac{\log \mathsf{P}}{\mathsf{L}}\right) + \frac{1}{2}\Upsilon. \end{aligned} \tag{40}$$

Taking the limit as SNR tends to infinity yields

$$\begin{aligned} &\lim_{\mathrm{SNR}\to\infty} C(\mathrm{SNR}) \\ &\quad \geq \lim_{\mathrm{SNR}\to\infty} \frac{1}{2}\log\left(\frac{\log(\mathrm{SNR}\cdot\sigma^2)}{\frac{\log(\mathrm{SNR}\cdot\varrho/(1-\varrho))}{\log(1/\varrho)}}\right) + \frac{1}{2}\Upsilon \\ &\quad = \frac{1}{2}\log\log\frac{1}{\varrho} + \frac{1}{2}\Upsilon. \end{aligned} \tag{41}$$

As this can be made arbitrarily large by choosing ϱ sufficiently small, we conclude that

$$\lim_{\ell\to\infty} \frac{1}{\ell}\log\frac{1}{\alpha_\ell} = \infty$$

implies that $C(\mathrm{SNR})$ is unbounded in SNR.

3. SUMMARY

We studied the capacity of discrete-time, noncoherent, multipath fading channels. It was shown that if the variances of the path gains decay faster than exponentially, then capacity is unbounded in the SNR. This complements previous results obtained in [3] and [4].

The overall picture looks as follows:

- If the number of paths is infinite in the sense that the channel output is influenced by the present and by *all* previous channel inputs, and if the variances of the path gains decay exponentially or slower, then capacity is bounded even as the SNR grows without bound.
- If the number of paths is infinite but the variances of the path gains decay faster than exponentially, then capacity tends to infinity as SNR $\to \infty$.
- If the number of paths is finite, then, irrespective of the number of paths, the capacity pre-loglog is 1. Thus, in this case the multipath behavior has no significant effect on the high-SNR capacity.

We thus see that the high-SNR behavior of the capacity of noncoherent multipath fading channels depends critically on the assumed channel model. Consequently, when studying such channels at high SNR, the channel modeling is crucial, as slight changes in the model might lead to completely different capacity results.

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