Teaching IT Power Spectral Density of Communication Signals

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Abstract

The transmitted waveforms in digital communications are rarely stationary, so they do not have a power spectral density (PSD) in the classical sense. To teach their PSD one needs a definition that is both general and useful. The traditional approach is to define the PSD via the average autocovariance function. Here I shall describe an alternative approach and offer some comparisons.

1 Introduction

The transmitted waveform in digital communications is usually modelled as a stochastic process (SP), because the data it conveys are viewed as random. But this SP is typically not wide-sense stationary (WSS), so the classical definition of the power spectral density (PSD) of a WSS SP as an integrable function whose Inverse Fourier Transform (IFT) is the SP's autocovariance function does not apply.

To overcome this difficulty, teachers often "stationarize" the signals in various ways. For example, in Pulse Amplitude Modulation (PAM), which is typically cyclostationary, they stationarize the random signal by introducing a random time offset. For Quadrature Amplitude Modulation (QAM) such an offset does not always suffice, and they thus also introduce a random phase. Once the process has been stationarized, they then apply the classical definition.

This approach has two shortcomings. The first is the lack of generality: different hacks are required to stationarize different transmission schemes. For example, in PAM the distribution of the time offset depends on whether or not forward error correction in the form of block coding is performed. And in QAM the need to introduce a random phase depends on whether or not the random sequence of complex symbols is proper. The second shortcoming is that this approach obscures the operational meaning of the PSD. Except for enabling them to calculate it on the exam, it is not clear to the students why knowing the PSD is useful. And saying that the Federal Communications Commission (FCC) places restrictions on it only begs the question as to why the FCC does so.

In the first part of this paper (Sections 1–4) I shall present a different approach, which I believe addresses these shortcomings. To avoid confusion with the classical PSD of WSS SPs, I shall refer to the PSD that I define as *Operational PSD* (OPSD). In the second part of the paper (Sections 5–6) I shall relate the OPSD to the *average autocovariance function*, which is often used to study nonstationary SPs [3, Ch. 4, Sec. 26.6]. The paper concludes with a discussion (Section 7) and some additional resources (Section 8).

To see the forest for the trees, I shall be somewhat informal and refer the interested readers to [2] for the technical details. In particular all the functions and SPs I consider are tacitly assumed measurable, and all the properties attributed to OPSD should be appended by the phrase "outside a set of frequencies of Lebesgue measure zero." Thus, when I write that the OPSD is "unique" I mean that two OPSDs of the same SP must be identical outside a set of frequencies of Lebesgue measure zero. A similar qualification applies when I say that the OPSD is "nonnegative." Also, to avoid unnecessary technical complications, we shall restrict attention to SPs of *bounded variance*, where a SP $\big(X(t)\big)$ is said to be of bounded variance if there exists some constant γ such that at every epoch $t \in \mathbb{R}$ the variance of the random variable (RV) $X(t)$ is bounded by γ:

$$\mathsf{Var}[X(t)] \le \gamma, \quad t \in \mathbb{R}. \tag{1}$$

Finally, we shall restrict ourselves to *centered stochastic processes*, i.e., to SPs of zero mean. The extensions to the general case are straightforward.

2 Power

We begin with the **power**, which is more intuitive and more fundamental.[1] The power in a SP $\big(X(t), t \in \mathbb{R}\big)$, or $\big(X(t)\big)$ or $\mathbf{X}$ for short, is P if

$$\mathsf{P} = \lim_{\mathsf{T}\to\infty} \frac{1}{2\mathsf{T}} \mathsf{E}\left[\int_{-\mathsf{T}}^{\mathsf{T}} X^2(t)\,\mathrm{d}t\right]. \tag{2}$$

For example, consider the PAM signal

$$X(t) = \mathsf{A} \sum_{\ell=-\infty}^{\infty} X_\ell \, g(t - \ell \mathsf{T}_\mathrm{s}), \quad t \in \mathbb{R}, \tag{3}$$

where $\mathsf{A}, \mathsf{T}_\mathrm{s} > 0$ are constants; the pulse-shape $\mathbf{g}$ is a deterministic real signal that decays sufficiently fast; and where the bi-infinite sequence $\ldots, X_{-1}, X_0, X_1, \ldots$ is bounded, centered, with

$$\mathsf{E}[X_\ell X_{\ell+m}] = \mathsf{K}_{XX}(m), \quad \ell, m \in \mathbb{Z}. \tag{4}$$

In this case a direct calculation [2, Section 14.5] shows that for any $\tau \in \mathbb{R}$,

$$\begin{aligned} &\mathsf{E}\left[\int_{\tau}^{\tau+\mathsf{T}_\mathrm{s}} X^2(t)\,\mathrm{d}t\right] \\ &\quad = \mathsf{A}^2 \sum_{m=-\infty}^{\infty} \mathsf{K}_{XX}(m) \int_{-\infty}^{\infty} g(t')\, g(t' - m\mathsf{T}_\mathrm{s})\,\mathrm{d}t' \\ &\quad = \mathsf{A}^2 \sum_{m=-\infty}^{\infty} \mathsf{K}_{XX}(m) \mathsf{R}_{\mathbf{gg}}(m\mathsf{T}_\mathrm{s}), \quad \tau \in \mathbb{R}, \end{aligned} \tag{5}$$

where $\mathsf{R}_{\mathbf{gg}}$ denotes the self-similarity function of $g(\cdot)$

$$\mathsf{R}_{\mathbf{gg}} : \tau \mapsto \int_{-\infty}^{\infty} g(t+\tau)\, g^*(t)\,\mathrm{d}t, \quad \tau \in \mathbb{R}. \tag{6}$$

From (5) we obtain

$$\begin{aligned} \left\lfloor \frac{2\mathsf{T}}{\mathsf{T}_\mathrm{s}} \right\rfloor \mathsf{E}\left[\int_{\tau}^{\tau+\mathsf{T}_\mathrm{s}} X^2(t)\,\mathrm{d}t\right] &\le \mathsf{E}\left[\int_{-\mathsf{T}}^{\mathsf{T}} X^2(t)\,\mathrm{d}t\right] \\ &\le \left\lceil \frac{2\mathsf{T}}{\mathsf{T}_\mathrm{s}} \right\rceil \mathsf{E}\left[\int_{\tau}^{\tau+\mathsf{T}_\mathrm{s}} X^2(t)\,\mathrm{d}t\right], \end{aligned} \tag{7}$$

[1] Teaching the power spectral density first and then integrating it to obtain the power is pedagogically unappealing and mathematically dubious; see Section 6.

and, thus, using the Sandwich Theorem,

$$\mathsf{P} = \frac{1}{\mathsf{T_s}}\,\mathsf{A}^2 \sum_{m=-\infty}^{\infty} \mathsf{K}_{XX}(m)\mathsf{R}_{\mathbf{gg}}(m\mathsf{T_s}) \tag{8}$$

$$= \frac{\mathsf{A}^2}{\mathsf{T_s}} \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \mathsf{K}_{XX}(m)\, e^{i2\pi f m \mathsf{T_s}}\, |\hat{g}(f)|^2 \,\mathrm{d}f, \tag{9}$$

where $\hat{\mathbf{g}}$ denotes the Fourier Transform (FT) of $g(\cdot)$:

$$\hat{g}(f) = \int_{-\infty}^{\infty} g(t)e^{-i2\pi f t}\,\mathrm{d}t, \quad f \in \mathbb{R}. \tag{10}$$

Computing the power in QAM is a bit trickier: the key is to relate the power in the QAM signal to the power in its baseband representation (which is a complex PAM signal) [2, Chapter 18].

3 Defining the OPSD

Denoting by $\mathcal{L}_1$ the class of real-valued functions from the reals whose Lebesgue integral is finite, we propose the following definition for the OPSD.

Definition 1 (Operational PSD of a Real SP). *We say that the continuous-time real stochastic process $\big(X(t), t \in \mathbb{R}\big)$ is of* ***operational power spectral density*** S_{XX} *if $\big(X(t), t \in \mathbb{R}\big)$ is a measurable SP; the mapping $\mathsf{S}_{XX} : \mathbb{R} \to \mathbb{R}$ is integrable and symmetric; and for every stable real filter of impulse response $\mathbf{h} \in \mathcal{L}_1$ the power at the filter's output when it is fed $\big(X(t), t \in \mathbb{R}\big)$ is given by*

$$\text{Power in } \mathbf{X} \star \mathbf{h} = \int_{-\infty}^{\infty} \mathsf{S}_{XX}(f)|\hat{\mathrm{h}}(f)|^2 \,\mathrm{d}f.$$

This functional relationship can be motivated by thinking of the power as being the sum of the powers in the infinitesimal nonoverlapping (and hence orthogonal) frequency slivers that the signal occupies. The symmetry requirement is only needed if we do not allow for complex filters. (The OPSD for complex SPs has the same definition except that the symmetry requirement is dropped and the filters are allowed to be complex.)

To put the reader at ease we note that, when it exists, the OPSD is "unique" [2, Corollary 15.3.3], and it is "nonnegative" [2, Exercise 15.5]. Moreover, for WSS SPs this definition coincides with the standard definition of the PSD as an integrable function whose IFT is the autocovariance function [2, Theorem 25.14.3].

Our definition makes it clear that knowing the OPSD of the transmitted waveform can be useful. For example, it allows us to calculate the "adjacent channel interference," i.e., how much of the signal's power "spills over" into the front-end filter of a receiver operating at an adjacent channel. Alas, it tells us nothing about how to compute the OPSD. This is, of course, the price of a general definition that must be applicable to a wide-range of transmission schemes.

As we shall see in Section 5, the OPSD can often be calculated from the *average autocovariance function* when the latter exists. However, it turns out that, for some of the transmission schemes that are taught in a basic course on digital communications, the OPSD can easily be calculated from its definition. Consider, for example, the PAM signal (3). Passing $\big(X(t)\big)$ through a stable filter of impulse response $\mathbf{h} \in \mathcal{L}_1$ is tantamount to replacing its pulse-shape $\mathbf{g}$ by $\mathbf{g} \star \mathbf{h}$ [2, Section 15.4], so the power in $\mathbf{X} \star \mathbf{h}$ can be calculated from (9) by replacing the FT of $\mathbf{g}$ with the FT of $\mathbf{g} \star \mathbf{h}$ to yield

$$\text{Power in } \mathbf{X} \star \mathbf{h} = \int_{-\infty}^{\infty} \Big(\frac{\mathsf{A}^2}{\mathsf{T_s}} \sum_{m=-\infty}^{\infty} \mathsf{K}_{XX}(m) \cdot e^{i2\pi f m \mathsf{T_s}}\, |\hat{g}(f)|^2\Big)\, |\hat{h}(f)|^2 \,\mathrm{d}f.$$

And since the term in parentheses is a symmetric function of f, it must coincide with the OPSD, so

$$\mathsf{S}_{XX}(f) = \frac{\mathsf{A}^2}{\mathsf{T_s}} \sum_{m=-\infty}^{\infty} \mathsf{K}_{XX}(m)\, e^{i2\pi f m \mathsf{T_s}}\, |\hat{g}(f)|^2. \tag{11}$$

The calculation of the OPSD for QAM signals can be carried out in a similar, albeit a bit more complicated, way [2, Section 18.4]. The key is to study the baseband representation of $\mathbf{X} \star \mathbf{h}$; to show that it corresponds to the filtering of the baseband representation of $\mathbf{X}$ (which is a complex PAM signal) by a (different) filter; and to then use the relationship between the power in baseband and passband.

4 The OPSD of a Filtered SP

Starting from the definition of the OPSD, it is rather simple to show that feeding a SP $\big(X(t)\big)$ of a given OPSD S_{XX} to a stable filter of a given impulse response $\mathbf{r} \in \mathcal{L}_1$ results in a SP of OPSD

$$f \mapsto \mathsf{S}_{XX}(f)\; |\hat{r}(f)|^2. \tag{12}$$

To see this only requires the small leap of faith that the associativity of the convolution extends to stochastic processes. Indeed, to compute the OPSD of $\mathbf{X} \star \mathbf{r}$ we need to know the power in $(\mathbf{X} \star \mathbf{r}) \star \mathbf{h}$ for every $\mathbf{h} \in \mathcal{L}_1$. But, since convolution is (usually) associative, we expect that the SP $(\mathbf{X} \star \mathbf{r}) \star \mathbf{h}$ be (usually) identical to the SP $\mathbf{X} \star (\mathbf{r} \star \mathbf{h})$ and hence of equal power. The power in the latter is easily computed from S_{XX}: we view $\mathbf{r} \star \mathbf{h}$ as an impulse response of a filter; we view $\mathbf{X} \star (\mathbf{r} \star \mathbf{h})$ as the result of passing $\mathbf{X}$ through this filter; and we recall that $\mathbf{X}$ is of OPSD S_{XX} so the power in $\mathbf{X} \star (\mathbf{r} \star \mathbf{h})$—and hence also in $(\mathbf{X} \star \mathbf{r}) \star \mathbf{h}$—is

$$\int_{-\infty}^{\infty} \mathsf{S}_{XX}(f)\; \left|\hat{r}(f)\hat{h}(f)\right|^2 \mathrm{d}f.$$

Rewriting this as

$$\int_{-\infty}^{\infty} \big(\mathsf{S}_{XX}(f)\; |\hat{r}(f)|^2\big)\; \left|\hat{h}(f)\right|^2 \mathrm{d}f,$$

and noting that the term in parentheses is symmetric in f, we conclude that the operational PSD of $\mathbf{X} \star \mathbf{r}$ should be given by (12).

5 The OPSD and the Average Autocovariance Function

We next explore the relationship between the OPSD and the **average autocovariance function**, which is defined as follows [3, Chapter 4, Section 26.6]:

Definition 2 (Average Autocovariance Function)**.** *We say that a SP $\big(X(t)\big)$ is of* ***average autocovariance function*** $\bar{\mathsf{K}}_{XX} : \mathbb{R} \to \mathbb{R}$ *if it is measurable, of bounded variance, and if for every $\tau \in \mathbb{R}$*

$$\lim_{\mathsf{T}\to\infty} \frac{1}{2\mathsf{T}} \int_{-\mathsf{T}}^{\mathsf{T}} \mathsf{Cov}[X(t), X(t+\tau)]\,\mathrm{d}t = \bar{\mathsf{K}}_{XX}(\tau). \tag{13}$$

By substituting 0 for τ in (13) and by recalling the definition of power (2), we obtain that if $\big(X(t)\big)$ is a centered SP of power P and of average autocovariance function $\bar{\mathsf{K}}_{XX}$, then

$$\mathsf{P} = \bar{\mathsf{K}}_{XX}(0). \tag{14}$$

An example of a SP that has an average autocovariance function is the PAM signal (3). In fact, the calculation of its average autocovariance function is very similar to the calculation of its power.

The following theorem provides an operational meaning to the average autocovariance function and shows that if it is integrable, then its FT is the OPSD. Thus, for stochastic processes having an integrable average autocovariance function, our definition of the OPSD and the definition in the literature of the operational PSD as the FT of $\bar{\mathsf{K}}_{XX}$ coincide.[2] It also provides a method for computing the operational PSD: compute $\bar{\mathsf{K}}_{XX}$ and take its FT.

Theorem 1 (The OPSD and the Average Autocovariance Function). *Let $\big(X(t)\big)$ be a centered SP of average autocovariance function $\bar{\mathsf{K}}_{XX}$.*

1) *If $\mathbf{h}$ is the impulse response of some stable filter, then*

$$\text{Power in } \mathbf{X} \star \mathbf{h} = \int_{-\infty}^{\infty} \bar{\mathsf{K}}_{XX}(\sigma)\, \mathsf{R}_{\mathbf{hh}}(\sigma)\, \mathrm{d}\sigma. \tag{15}$$

2) *If $\bar{\mathsf{K}}_{XX}$ is integrable, then its Fourier Transform is the OPSD of $\big(X(t)\big)$:*

$$\hat{\bar{\mathsf{K}}}_{XX} = \mathsf{S}_{XX}. \tag{16}$$

6 The OPSD and Power

Intuition suggests that the OPSD should integrate to the power. To see why, recall that if $\mathbf{X}$ is of OPSD S_{XX}, then

$$\text{Power in } \mathbf{X} \star \mathbf{h} = \int_{-\infty}^{\infty} \mathsf{S}_{XX}(f)\, \left|\hat{h}(f)\right|^2 \mathrm{d}f, \quad \mathbf{h} \in \mathcal{L}_1. \tag{17}$$

Suppose we now substitute for $\mathbf{h}$ the impulse response of a filter whose frequency response resembles that of an ideal unit-gain lowpass filter of very large cutoff frequency $\mathsf{W} \gg 1$. In this case the RHS of (17) would resemble the integral of $\mathsf{S}_{XX}(f)$ from $-\mathsf{W}$ to $+\mathsf{W}$, which is approximately the integral from $-\infty$ to $+\infty$ when W is very large. And as to the LHS, if W is very large, then intuition suggests that $\mathbf{X}$ will hardly be altered by the filter, and the LHS would approximately equal the power in $\mathbf{X}$.

This intuition is excellent, and for most stochastic processes of interest the OPSD indeed integrates to the power. However, as our next example shows, there are some pathological counter-examples. In fact, in the absence of additional assumptions, we are only guaranteed that the integral of the OPSD cannot exceed the power.

Before presenting our example in detail, we begin with the big picture. In our example the SP $\mathbf{X}$ takes on the values ± 1 only, so its power is 1. However, $\mathbf{X}$ changes between the values $+1$ and -1 progressively faster the further time is from the origin. As we next explain, this results in the power in $\mathbf{X} \star \mathbf{h}$ being zero for every stable filter $\mathbf{h}$, so $\mathbf{X}$ is of zero OPSD. The integral of the operational PSD is thus zero, while the power is one.

[2] The FT of the average autocovariance function is called "average spectral density" in [3].

For some intuition as to why the power in $\mathbf{X} \star \mathbf{h}$ is zero, recall that when $\mathbf{h}$ is stable, its frequency response decays to zero. Consequently, above some cutoff frequency, the frequency response of the filter is nearly zero. Since our SP varies faster and faster the further we are from the origin of time, when we are sufficiently far from the origin of time the dynamics of our SP are much faster than the filter's cutoff frequency. Consequently, except for transients that result from the behavior of our SP near the origin of time, in steady state the response of $\mathbf{h}$ to $\mathbf{X}$ will be nearly zero. Since the transients do not influence the power in $\mathbf{X} \star \mathbf{h}$, the power in $\mathbf{X} \star \mathbf{h}$ is zero. We next present the example in greater detail.

Example 1. *Consider the SP $\big(X(t), t \in \mathbb{R}\big)$ whose value in the time interval $[\nu, \nu+1)$ is defined for every integer ν as follows: The interval is divided into $|\nu|+1$ nonoverlapping half-open subintervals of length $1/(|\nu|+1)$*

$$\left[\nu + \frac{\kappa}{|\nu|+1}, \nu + \frac{\kappa+1}{|\nu|+1}\right), \quad \kappa \in \{0, \ldots, |\nu|\},$$

and in each such subinterval the SP is constant and is equal to the RV $X_{\nu,\kappa}$, which takes on the values ± 1 equiprobably with

$$\{X_{\nu,\kappa}\}, \quad \nu \in \mathbb{Z}, \kappa \in \{0, \ldots, |\nu|\}$$

being IID. Thus,

$$X(t) = \sum_{\nu=-\infty}^{\infty} \sum_{\kappa=0}^{|\nu|} X_{\nu,\kappa} \mathrm{I}\left\{\nu + \frac{\kappa}{|\nu|+1} \le t < \nu + \frac{\kappa+1}{|\nu|+1}\right\}, \tag{18a}$$

$$\{X_{\nu,\kappa}\} \sim \text{IID } \mathcal{U}(\{\pm 1\}). \tag{18b}$$

This SP is centered, of power $\mathsf{P} = 1$, and yet its operational PSD is zero at all frequencies. The integral of the OPSD of $\mathbf{X}$ is thus strictly smaller than the power in $\mathbf{X}$.

Proof. At every epoch t the RV $X(t)$ takes on the values ± 1 equiprobably and is thus centered. Moreover, $X^2(t)$ is deterministically 1, so the power in $\big(X(t)\big)$ is one. We next show that $\big(X(t)\big)$ is of average autocovariance function

$$\bar{\mathsf{K}}_{XX}(\tau) = \begin{cases} 1 & \text{if } \tau = 0, \\ 0 & \text{otherwise,} \end{cases} \quad \tau \in \mathbb{R}. \tag{19}$$

For τ equal to zero this follows immediately from our observation that $X^2(t)$ is deterministically equal to one. By symmetry, it suffices to establish (19) for positive τ. When τ is 1 or larger, the epochs t and $t + \tau$ fall—irrespective of t—in different intervals, so $X(t)$ and $X(t + \tau)$ are uncorrelated for all t. For such τ's $\bar{\mathsf{K}}_{XX}(\tau)$ is thus zero, in agreement with (19). It thus only remains to establish (19) for $0 < \tau < 1$. In this case t and $t + \tau$ are guaranteed to fall in different subintervals whenever

$$\tau \ge \frac{1}{|\lfloor t \rfloor|+1}, \tag{20}$$

where the RHS is the length of the subintervals to which the interval containing t—namely the interval $[\nu, \nu+1)$, where ν is $\lfloor t \rfloor$—is subdivided. (If this inequality is not satisfied, then $X(t)$ and $X(t + \tau)$ may or may not be in different subintervals.) For $\tau \in (0,1)$ Inequality (20) holds whenever $|\lfloor t \rfloor| \ge \tau^{-1} - 1$. Thus, when t is outside the finite interval

$$\left\{t' \in \mathbb{R} : \ |\lfloor t' \rfloor| < \tau^{-1} - 1\right\}$$

the random variables $X(t)$ and $X(t+\tau)$ are uncorrelated. For t inside this finite interval the correlation between $X(t)$ and $X(t+\tau)$ is upper bounded by 1. Consequently, when we average $\mathsf{E}[X(t)X(t+\tau)]$ over t, the contribution of t's inside this interval washes out and the result is zero.

From (19) we conclude using Theorem 1 (ii) that the OPSD of $\big(X(t)\big)$ is zero. □

In Example 1 the power is strictly larger than the integral of the OPSD, and the average autocovariance function is discontinuous at the origin. This is no coincidence: the integral of the OPSD never exceeds the power, and the two are the same whenever the SP has an average autocovariance function that is continuous at the origin:

Theorem 2 (The Power and the Integral of the OPSD). *Let* $\big(X(t)\big)$ *be a centered SP of OPSD* S_{XX} *and of power* P.

1) *The integral of the OPSD never exceeds the power*:

$$\mathsf{P} \geq \int_{-\infty}^{\infty} \mathsf{S}_{XX}(f)\,\mathrm{d}f. \tag{21}$$

2) *If, additionally,* $\big(X(t)\big)$ *is of some average autocovariance function* $\bar{\mathsf{K}}_{XX}$*, then equality in* (21) *holds if, and only if,* $\bar{\mathsf{K}}_{XX}$ *is continuous at the origin*.

7 Discussion

To teach the PSD we must provide the students with a general definition, an operational meaning, and some useful examples. Definition 1 provides the first two, and the class of PAM signals the third. PAM signals are particularly suitable for this purpose because filtering a PAM signal is tantamount to filtering its pulse shape, so—once we have taught the power in PAM—we can easily also calculate the power in filtered PAM. Another example is provided by QAM signals, but the analysis is a bit more difficult. Note, however, that this additional difficulty is already encountered in the calculation of the power, and, once we have taught the power, the OPSD is fairly straightforward.

A different viable approach is to define the PSD as the FT of the average autocovariance function. But if this approach is adopted, then one must also provide the students with an operational meaning such as that of Theorem 1(i). Once again, PAM signals can provide the desired example, but QAM might be a bit trickier.

The drawback of Definition 1 is that it is not immediately obvious from the definition that the OPSD is "unique" [2, Corollary 15.3.3]. But the added benefit is that it makes it almost obvious how the OPSD should behave when the SP is filtered (Section 4). At the end of the day it is up to the instructor to decide which definition is preferable. I prefer Definition 1 because providing the operational meaning to the FT of the average autocovariance function (Theorem 1(i)) requires a significant detour, and because Definition 1 is particularly suitable not only for PAM but also for QAM.

I am not very keen on teaching the OPSD by stationarizing the SP and by then using the classical definition for WSS SPs. This approach lacks generality and obscures the operational meaning. Moreover, in QAM it hides the beautiful result that the OPSD does not depend on the pseudo-covariance of the symbols. Indeed, this approach introduces a random phase that is tantamount to setting the pseudo-autocovariance function to zero and making the symbols proper.

Some readers who are familiar with the workings of a spectrum analyzer might contemplate using that as a pedagogical tool for teaching the OPSD. I suspect, however, that this might lead to confusion because in a spectrum analyzer time-averages and ensemble-averages are intertwined.[3] Moreover, different spectrum analyzers work in different ways and thus lead to different possible definitions. Some measure the power at the output of narrow bandpass filters centered around the different frequencies while others use the FFT. Moreover, the order in which the different limits are taken when analyzing a nonstationary SP using a spectrum analyzer is tricky. A related approach, which some teachers use to motivate the PSD of WSS SPs, is to study the limit

$$\lim_{\mathsf{T}\to\infty} \mathsf{E}\left[\left|\frac{1}{\sqrt{2\mathsf{T}}}\int_{-\mathsf{T}}^{\mathsf{T}} X(t)e^{-i2\pi ft}\,\mathrm{d}t\right|^2\right].$$

But relating this limit to the FT of the average autocovariance function can be tricky.

The OPSD is not only important in applications, but also a pleasure to teach. Whether you adopt Definition 1 is immaterial: what is important is that you go out and teach *it*.

8 Additional Resources

Most of the material on the OPSD can be found in the textbook [2] and in the videos of my lectures, which can be found at

`http://www.multimedia.ethz.ch/lectures/itet/2013/spring/227-0104-00L/`

Chapter 14, which is presented in Lecture 6, defines power and computes it for PAM; Chapter 15, which is presented in Lecture 7, defines the OPSD and computes it for PAM; and Chapter 18, which is presented in Lecture 9, computes the power and OPSD for QAM signals and also defines the OPSD for complex SPs. A shorter video on the OPSD of QAM can be found at

`http://www.afidc.ethz.ch/A_Foundation_in_Digital_Communication/QAMMovie.html`

The video emphasizes that the OPSD of QAM does not depend on the pseudo-covariance of the transmitted symbols.

An excellent starting point for the literature on the average autocovariance function is Note 174 in [4]. And for more on cyclostationarity see [1].

References

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[5] A.M. Yaglom, "Einstein's 1914 paper on the theory of irregularly fluctuating series of observations," *IEEE ASSP Magazine*, pp. 7–11, Oct. 1987.

[3] For an excellent historical account of this issue, see [5]. (Thank you S. Verdu.)