2016 ICSEE International Conference on the Science of Electrical Engineering

A Note on Feedback Communication with Mismatched Decoding

Amos Lapidoth
ETH Zurich

***Abstract*—It is shown that, even on memoryless channels, feedback may be beneficial when the decoder is constrained to use a given suboptimal decoding metric. In fact, subject to certain conditions that we formulate, feedback allows reliable communication at rates approaching Shannon's capacity. For scenarios where these conditions are not satisfied, we propose to exploit the feedback using a two-phase coding scheme.**

I. Introduction

The mismatch capacity—the highest rate at which reliable communication is possible with a given decoding rule—has been studied extensively over the years; see [1]–[7] and references therein. Here we draw on recent results on the erasures-only capacity (aka the zero-undetected-error capacity) with feedback [8], [9] and on Csiszár and Narayan's observation that the erasures-only capacity is a special case of the mismatch capacity [4] to infer that feedback can increase the mismatch capacity. With the coding scheme of [9] in mind, we derive a condition that guarantees that the mismatch capacity with feedback equal the channel's Shannon capacity. For scenarios where this condition is not satisfied, we propose to exploit the feedback using a two-phase scheme, which provides a lower bound on the feedback capacity of discrete memoryless channels with a given single-letter decoding rule.

II. Problem Statement and Main Result

A discrete-memoryless channel (DMC) of transition law $W(y|x)$ over the finite input alphabet $\mathcal{X}$ and the finite output alphabet $\mathcal{Y}$ is given. Also given is a "decoding metric" $d(x,y)$, i.e., a mapping from $\mathcal{X}\times\mathcal{Y}$ to the extended nonnegatives $[0,\infty]$

$$d\colon \mathcal{X}\times\mathcal{Y}\to[0,\infty].$$

The decoding metric is extended to k-length sequences $\mathbf{x}=(x_1,\dots,x_k)$ and $\mathbf{y}=(y_1,\dots,y_k)$ additively:

$$d(\mathbf{x},\mathbf{y})=\sum_{j=1}^{k} d\bigl(x_j,y_j\bigr). \tag{1}$$

A blocklength-n codebook $\mathcal{C}$ for M messages maps each message m in the message set $\mathcal{M}=\{1,\dots,\mathsf{M}\}$ to an n-tuple

$$\mathbf{x}(m)=\bigl(x_1(m),\dots,x_n(m)\bigr)\in\mathcal{X}^n.$$

Upon observing the output sequence $\mathbf{y}$, the d-based decoder produces the message accumulating the least metric, i.e., the message m for which

$$d\bigl(\mathbf{x}(m),\mathbf{y}\bigr)<d\bigl(\mathbf{x}(m'),\mathbf{y}\bigr),\quad m'\neq m.$$

If no such $m\in\mathcal{M}$ exists, i.e., if the minimum over m of $d\bigl(\mathbf{x}(m),\mathbf{y}\bigr)$ is achieved by more than one message, then an error is declared. We refer to $d(\mathbf{x}(m),\mathbf{y})$ as the metric accumulated by the m-th message, and we denote the output of the d-based decoder by $\phi_d(\mathbf{y})$ or $\phi_d(\mathbf{y};\mathcal{C})$. When $d(x,y)$ equals $-\log W(y|x)$, the d-based decoder reduces to the maximum-likelihood decoder ϕ_{ML}, which produces $\phi_{\mathrm{ML}}(\mathbf{y})$ or $\phi_{\mathrm{ML}}(\mathbf{y};\mathcal{C})$. A rate R is achievable (without feedback) over the channel $W(y|x)$ using the decoding metric $d(x,y)$ if, for every $\epsilon>0$, there exists some sufficiently-large blocklength n_0, such that for every blocklength n exceeding n_0 we can find a codebook of at least 2^{nR} codewords for which the d-based decoder errs with probability smaller than ϵ (irrespective of the transmitted message). The supremum of all achievable rates is the *mismatch capacity*, which we denote by $C(W,d)$.

When a feedback link is present, a blocklength-n encoder is a collection of n mappings $f_1,\dots,f_n$, where

$$f_i\colon \mathcal{M}\times\mathcal{Y}^{i-1}\to\mathcal{X},$$

with the understanding that $f_i(m,y^{i-1})$—which we also denote $x_i\bigl(m,y^{i-1}\bigr)$—is the symbol that the encoder transmits at Time i in order to convey Message m after having observed the $i-1$ outputs $y^{i-1}=(y_1,\dots,y_{i-1})$. The d-based decoder produces the message m for which

$$\sum_{j=1}^{n} d\Bigl(x\bigl(m,y^{j-1}\bigr),y_j\Bigr)<\sum_{j=1}^{n} d\Bigl(x\bigl(m',y^{j-1}\bigr),y_j\Bigr),\quad m'\neq m.$$

When we allow for feedback we speak of "an achievable rate with feedback," and we define the *mismatch capacity with feedback* as the supremum of the rates that are achievable with feedback. We denote this capacity $C_{\mathrm{FB}}(W,d)$. The *Shannon capacity*, i.e., the maximum of $I(X;Y)$ over all input distributions, is denoted $C_{\mathrm{Sh}}(W)$. Since restricting the decoding rule cannot help,

$$C_{\mathrm{FB}}(W,d)\le C_{\mathrm{Sh}}(W). \tag{2}$$

We now formulate the conditions that will guarantee that $C_{\mathrm{FB}}(W,d)$ equal $C_{\mathrm{Sh}}(W)$. The first is that

$$\Bigl(W(y|x)>0\Bigr)\implies\Bigl(d(x,y)<\infty\Bigr). \tag{3a}$$

It guarantees that the Maximum-Likelihood message is not ruled out by the d-based decoder: the metric it accumulates is, with probability one, finite. (See also the lead-up to (15) ahead.)

978-1-5090-2152-9/16/$31.00 ©2016 IEEE

We shall also require that (W, d) be usable as a Z-channel in the sense that there exist two inputs $x_{\mathrm{T}}, x_{\mathrm{F}} \in \mathcal{X}$ and a collection of output symbols $\mathcal{Y}_{\mathrm{T}} \subset \mathcal{Y}$ such that

$$\delta \triangleq \Pr[Y \notin \mathcal{Y}_{\mathrm{T}} \,|\, X = x_{\mathrm{T}}] < 1, \tag{3b}$$

$$d\big(x_{\mathrm{F}}, y\big) = \infty, \quad y \in \mathcal{Y}_{\mathrm{T}}, \tag{3c}$$

$$d\big(x_{\mathrm{T}}, y\big) < \infty, \quad y \in \mathcal{Y}_{\mathrm{T}}. \tag{3d}$$

Note that (3a) and (3c) imply that if $W(y|x_{\mathrm{F}})$ is positive then y cannot be in $\mathcal{Y}_{\mathrm{T}}$. Thus,

$$\Pr\big[Y \in \mathcal{Y}_{\mathrm{T}} \,\big|\, X = x_{\mathrm{F}}\big] = 0. \tag{4}$$

It is (3b) and (4) that remind us of the Z-channel, and it is (3c) and (3d) that indicate that $d(x, y)$ allows us to use the channel as a Z-channel.

Theorem II.1. *If the channel $W(y|x)$ and the decoding metric $d(x, y)$ are such that there exist inputs $x_{\mathrm{T}}, x_{\mathrm{F}} \in \mathcal{X}$ and a subset $\mathcal{Y}_{\mathrm{T}}$ of the output alphabet $\mathcal{Y}$ such that* (3) *hold, then*

$$C_{\mathrm{FB}}(W, d) = C_{\mathrm{Sh}}(W). \tag{5}$$

For multiple-access channels we can obtain an analogous result by building on the approach of [9, Section IV] in much the same way that we built Theorem II.1 on [9, Theorem 1].

When the hypotheses of Theorem II.1 are not satisfied, feedback can still be beneficial. This can be gleaned from [9, Secton III] on the erasures-only capacity, which can be posed as a mismatch capacity with a finite decoding metric. A coding scheme for this scenario is presented in Section V, and the achievable rates is promises are presented in Theorem V.1.

III. Proof of Theorem II.1

By (2), the proof of Theorem II.1 only requires a direct part, i.e., a coding scheme with feedback that achieves C_{Sh} with the d-based decoder. Given $\epsilon', \epsilon > 0$, we will construct, for every sufficiently-large blocklength n, a feedback scheme of rate exceeding $C_{\mathrm{Sh}} - \epsilon'$ for which the maximal probability of error incurred by the d-based decoder is smaller than ϵ.

Let η be a sufficiently-large positive integer so that

$$\delta^{\eta} < \frac{\epsilon}{2}, \tag{6}$$

where δ is defined in (3b). Given any M-codewords blocklength-n codebook $\mathbf{x}(1), \ldots, \mathbf{x}(\mathsf{M})$ *without feedback*, and any (e.g., maximum-likelihood) corresponding decoder $\phi\colon \mathcal{Y}^n \to \mathcal{M}$, we can construct an M-codewords blocklength-$(n + \eta)$ codebook *with feedback* as follows: To transmit Message m using $n+\eta$ channel uses, we send $\mathbf{x}(m)$ in the first n channels uses, followed by η channel inputs that are either the all-x_{T} inputs or the all-x_{F} inputs, depending on whether the first n outputs y^n are such that $\phi(y^n)$ is m or not. Thus, the first n channel inputs are

$$f_i\big(m, y^{i-1}\big) = x_i(m), \quad i = 1, \ldots, n, \tag{7a}$$

and the subsequent η channel inputs are

$$f_i\big(m, y^{i-1}\big) = \begin{cases} x_{\mathrm{T}} & \text{if } \phi\big(y^n\big) = m, \\ x_{\mathrm{F}} & \text{otherwise,} \end{cases} \quad i = n+1, \ldots, n+\eta. \tag{7b}$$

Suppose now that n is sufficiently large to guarantee the existence of a blocklength-n codebook $\mathbf{x}(1), \ldots \mathbf{x}(\mathsf{M})$ that nearly achieves Shannon's capacity in the sense that

$$\frac{1}{n} \log \mathsf{M} > C_{\mathrm{Sh}} - \frac{\epsilon'}{2} \tag{8}$$

and that, with some decoding rule ϕ (not necessarily ϕ_d) its maximal probability of error is smaller than $\epsilon/2$

$$\Pr\big[\phi(Y^n) \neq m \,\big|\, M = m\big] < \frac{\epsilon}{2}, \quad m \in \mathcal{M}. \tag{9}$$

Increasing the blocklength if needed, we can also assume that

$$\frac{n}{n + \eta} > 1 - \frac{\epsilon'}{2 C_{\mathrm{Sh}}}. \tag{10}$$

Construct now a blocklength-$(n+\eta)$ feedback scheme based on $\mathbf{x}(1) \ldots, \mathbf{x}(\mathsf{M})$ as in (7). By (10) and (8) the rate of the feedback scheme exceeds $C_{\mathrm{Sh}} - \epsilon'$:

$$\frac{1}{n + \eta} \log \mathsf{M} > C_{\mathrm{Sh}} - \epsilon'. \tag{11}$$

We next show that this feedback scheme—when decoded with the d-based decoding rule ϕ_d—has a small maximal probability of error. To that end, suppose that Message m is sent, and assume for now that $\phi(y^n) = m$. (This is the case with probability exceeding $1 - \epsilon/2$; see (9)). We will demonstrate that in this case, $\phi_d(Y^{n+\eta})$ equals m with (conditional) probability exceeding $1 - \delta^{\eta}$. We shall do so by noting that, by (3a), the metric accumulated by Message m is finite, and by showing that, with probability exceeding $1 - \delta^{\eta}$, the metrics accumulated by all the other messages are infinite.

In the case under consideration the transmitted message is m and $\phi(y^n) = m$, so $X_i = f_i(m, y^{i-1}) = x_{\mathrm{T}}$ for all $i = n+1, \ldots, n+\eta$. Consider now the event that at least one of the outputs $Y_{n+1}, \ldots, Y_{n+\eta}$ is in $\mathcal{Y}_{\mathrm{T}}$. The conditional probability of this event is $1 - \delta^{\eta}$ (3b). We claim that if this event occurs, then, in the case under consideration, all the messages other than m will accumulate an infinite metric. To see why, assume that this event occurred and let $n + \kappa$ be one among the times $n + 1, \ldots, n + \eta$ at which the output is an element of $\mathcal{Y}_{\mathrm{T}}$

$$Y_{n+\kappa} \in \mathcal{Y}_{\mathrm{T}}. \tag{12}$$

We will next show that, whenever $m' \neq m$, the contribution of the term

$$d\Big(f_{n+\kappa}\big(m', y^{n+\kappa-1}\big), y_{n+\kappa}\Big)$$

to the metric accumulated by m' is infinite. Indeed—since in the case under consideration $\phi(y^n) = m$, and m' differs from m—we have $f_i(m', y^{i-1}) = x_{\mathrm{F}}$ for every $i = n+1, \ldots, n+\eta$ and hence, *a-fortiori*, $f_{n+\kappa}(m', y^{n+\kappa-1}) = x_{\mathrm{F}}$. Consequently, by (3c), the contribution of this term to the metric accumulated by m' is indeed infinite. More formally,

$$\begin{aligned}
&\Pr\Big[\phi_d\big(Y^{n+\eta}\big) = m \,\Big|\, M = m\Big] \\
&\geq \Pr\Big[\phi_d\big(Y^{n+\eta}\big) = m \,\Big|\, \phi\big(Y^n\big) = m,\ M = m\Big] \\
&\quad \cdot \Pr\Big[\phi\big(Y^n\big) = m \,\Big|\, M = m\Big] \\
&\geq \Pr\Big[\sum_{j=n+1}^{n+\eta} \mathrm{I}\{Y_j \in \mathcal{Y}_{\mathrm{T}}\} \geq 1 \,\Big|\, X_{n+1} = \cdots = X_{n+\eta} = x_{\mathrm{T}}\Big] \\
&\quad \cdot \Pr\Big[\phi\big(Y^n\big) = m \,\Big|\, M = m\Big] \\
&= \big(1 - \delta^\eta\big) \Pr\Big[\phi\big(Y^n\big) = m \,\Big|\, M = m\Big] \\
&> \Big(1 - \frac{\epsilon}{2}\Big)^2 \\
&> 1 - \epsilon,
\end{aligned}$$

where the second inequality holds because, as we argued above, if $M = m$, if $\phi(Y^n) = m$, and if at least one of the outputs among $Y_{n+1}, \ldots, Y_{n+\eta}$ is in $\mathcal{Y}_{\mathrm{T}}$, then the d-based decoder does not err. The third inequality follows from (6) and (9). □

IV. Reduction to Finite Decoding Metrics

What can we say in case the hypotheses of Theorem II.1 do not hold? To address this case, we next argue that to fully characterize $C_{\mathrm{FB}}(W, d)$ it remains to treat the case of finite decoding metrics,

$$d(x, y) < \infty, \quad (x, y) \in \mathcal{X} \times \mathcal{Y}. \tag{13}$$

To see why, consider a channel where d is infinite, and denote by $\tilde{\epsilon}$ the smallest positive transition probability

$$\tilde{\epsilon} = \min\big\{W(y|x) : W(y|x) > 0,\ (x, y) \in \mathcal{X} \times \mathcal{Y}\big\}. \tag{14}$$

If we require a maximal probability of error smaller than $\tilde{\epsilon}$, then we should never use an input x for which there exists some $y \in \mathcal{Y}$ with $W(y|x) > 0$ and $d(x, y) = \infty$, because, when fed x, the channel might produce y with the result that the correct message will accumulate an infinite metric. We can thus discard all such inputs, and reduce the input alphabet to

$$\mathcal{X}' = \Big\{x \in \mathcal{X} : \big(d(x, y) = \infty\big) \implies \big(W(y|x) = 0\big)\Big\}. \tag{15}$$

After having gotten rid of all the useless inputs, we can reduce the output alphabet to those symbols that can be reached from at least one of the inputs in the reduced input alphabet $\mathcal{X}'$

$$\mathcal{Y}' = \Big\{y \in \mathcal{Y} : \exists x \in \mathcal{X}' \text{ s.t. } W(y|x) > 0\Big\}. \tag{16}$$

We now consider two cases separately, depending on whether or not d is finite on $\mathcal{X}' \times \mathcal{Y}'$. We argue that if it is not, then Theorem II.1 implies that the feedback mismatch capacity of the original channel—which equals that of the reduced channel—is equal to the reduced channel's Shannon capacity.

Indeed, if d is not finite on $\mathcal{X}' \times \mathcal{Y}'$, then there exist $x' \in \mathcal{X}'$ and $y' \in \mathcal{Y}'$ such that

$$d(x', y') = +\infty. \tag{17a}$$

From the definition of $\mathcal{X}'$ (15) it then follows that

$$W(y'|x') = 0. \tag{17b}$$

And, from the definition of $\mathcal{Y}'$ (16), y' must be reachable from some input symbol from $\mathcal{X}'$, which cannot be x' by (17b). Thus, there exists some $x'' \in \mathcal{X}' \setminus \{x'\}$ such that

$$W(y'|x'') > 0. \tag{17c}$$

This and the fact that x'' is in $\mathcal{X}'$ imply by (15) that

$$d(x'', y') < \infty. \tag{17d}$$

We can now apply Theorem II.1 with

$$x_{\mathrm{F}} = x' \quad x_{\mathrm{T}} = x'' \quad \mathcal{Y}_{\mathrm{T}} = \{y'\} \tag{18}$$

to infer that, when d is infinite on $\mathcal{X}' \times \mathcal{Y}'$, the feedback mismatch capacity of the reduced channel is equal to its Shannon capacity.

V. A General Two-Phase Scheme

At present we do not, alas, have an expression for the mismatch feedback capacity for finite decoding metrics. We do, however, have a lower bound, which is motivated by the proof of Theorem II.1. In view of Section IV, we now focus only on finite decoding metrics (13).

The scheme we propose transmits two messages, m_1 and m_2, in two phases. In Phase 1, Message m_1 is transmitted in n_1 channel uses using a rate-R_1 blocklength-n_1 codebook $\mathcal{C}_1$ without utilizing the feedback link. We denote the transmitted n_1 channel inputs, i.e., the m_1-th codeword in $\mathcal{C}_1$, by $\mathbf{x}^{(1)}(m_1)$. The salient requirments from $\mathcal{C}_1$ are a small maximal probability of error under maximum-likelihood decoding, and a guarantee that with high probability the d-metric accumulated by the incorrect codewords be above some threshold (later specified as $n_1 \Lambda_{1,\mathrm{w}}$). The codebook $\mathcal{C}_1$ will be constructed using random-coding based on some input distribution Q_1.[1]

Let $\mathbf{y}^{(1)}$ denote the channel outputs corresponding to Phase 1, i.e., the first n_1 channel outputs $y_1, \ldots, y_{n_1}$. Let ϕ_{ML} be the maximum-likelihood (or the joint-typicality) decoder for the code $\mathcal{C}_1$. The transmission in Phase 2 depends on whether or not $\phi_{\mathrm{ML}}\big(\mathbf{y}^{(1)}; \mathcal{C}_1\big)$ equals m_1. If not, then in Phase 2 we transmit the input symbol $x_{\mathrm{F}} \in \mathcal{X}$ repeatedly n_2 times. If yes, then in Phase 2 we send the second message m_2 using a blocklength-n_2 rate-R_2 codebook $\mathcal{C}_2$. The salient requirements from $\mathcal{C}_2$ are that it be reliably decodable using the mismatched decoder ϕ_d, and that, with high probability,

[1]The codewords are chosen independently, with each codeword chosen uniformly over a type class that closely resembles Q_1 or according to a conditional distribution over a "shell" around Q_1 as in [3].

the decoding metric associated with the all-x_{F} sequence be higher than some threshold (later specified as $n_2\,\Lambda_{\mathrm{F}}$).

Denoting the all-x_{F} n_2-tuple by $\mathbf{x}_{\mathrm{F}}$

$$\mathbf{x}_{\mathrm{F}} = \underbrace{x_{\mathrm{F}}, \ldots, x_{\mathrm{F}}}_{n_2 \text{ times}} \in \mathcal{X}^{n_2}, \tag{19}$$

and the m_2-th codeword in $\mathcal{C}_2$ by $\mathbf{x}^{(2)}(m_2)$, we can express the transmitted n_2-tuple in Phase 2 as

$$\begin{cases} \mathbf{x}_{\mathrm{F}} & \text{if } \phi_{\mathrm{ML}}\big(\mathbf{y}^{(1)}; \mathcal{C}_1\big) \neq m_1, \\ \mathbf{x}^{(2)}(m_2) & \text{otherwise.} \end{cases}$$

Denoting by $\mathbf{x}\big(m_1, m_2, \mathbf{y}^{(1)}\big)$ the transmitted n-tuple when the messages to be transmitted are (m_1, m_2) and the n_1 channel outputs corresponding to Phase 1 are $\mathbf{y}^{(1)}$,

$$\begin{aligned} &\mathbf{x}\big(m_1, m_2, \mathbf{y}^{(1)}\big) \\ &\quad = \begin{cases} \big(\mathbf{x}^{(1)}(m_1), \mathbf{x}_{\mathrm{F}}\big) & \text{if } \phi_{\mathrm{ML}}\big(\mathbf{y}^{(1)}; \mathcal{C}_1\big) \neq m_1, \\ \big(\mathbf{x}^{(1)}(m_1), \mathbf{x}^{(2)}(m_2)\big) & \text{otherwise.} \end{cases} \end{aligned} \tag{20}$$

Defining

$$\alpha = \frac{n_1}{n_1 + n_2}, \tag{21}$$

the transmission rate R of the two-phase scheme is

$$R = \alpha R_1 + (1 - \alpha) R_2. \tag{22}$$

Given the output $\mathbf{y} = \big(\mathbf{y}^{(1)}, \mathbf{y}^{(2)}\big)$, the mismatched decoder ϕ_d, produces the pair (m_1, m_2) if, and only if,

$$\begin{aligned} d\big(\mathbf{x}(m_1, m_2, \mathbf{y}^{(1)}), \mathbf{y}\big) < d\big(\mathbf{x}(m_1', m_2', \mathbf{y}^{(1)}), \mathbf{y}\big), \\ (m_1', m_2') \neq (m_1, m_2), \end{aligned}$$

where for every pair of messages (m_1', m_2'),

$$\begin{aligned} &d\big(\mathbf{x}(m_1', m_2', \mathbf{y}^{(1)}), \mathbf{y}\big) \\ &\quad = \begin{cases} d\big(\mathbf{x}^{(1)}(m_1'), \mathbf{y}^{(1)}\big) + d\big(\mathbf{x}^{(2)}(m_2'), \mathbf{y}^{(2)}\big) \\ \qquad\qquad \text{if } \phi_{\mathrm{ML}}\big(\mathbf{y}^{(1)}; \mathcal{C}_1\big) = m_1', \\ d\big(\mathbf{x}^{(1)}(m_1'), \mathbf{y}^{(1)}\big) + d\big(\mathbf{x}_{\mathrm{F}}, \mathbf{y}^{(2)}\big) \\ \qquad\qquad \text{otherwise.} \end{cases} \end{aligned} \tag{23}$$

To specify the requirements from the codebook $\mathcal{C}_1$, let $\mathcal{E}_{m_1}(\Lambda_{1,\mathrm{c}}, \Lambda_{1,\mathrm{w}}) \subseteq \mathcal{Y}^{n_1}$ comprise the Phase-1 outputs $\mathbf{y}^{(1)}$ for which $\phi_{\mathrm{ML}}(\mathbf{y}^{(1)}; \mathcal{C}_1)$ differs from m_1, or for which Message m_1 accumulates a metric greater-equal $n_1 \Lambda_{1,\mathrm{c}}$, or that result in some message m_1' other than m_1 accumulating a metric lower than $n_1\,\Lambda_{1,\mathrm{w}}$,

$$\begin{aligned} &\mathcal{E}_{m_1}\big(\Lambda_{1,\mathrm{c}}, \Lambda_{1,\mathrm{w}}\big) \\ &\quad = \Big\{ \mathbf{y}^{(1)} \in \mathcal{Y}^{n_1} : \phi_{\mathrm{ML}}\big(\mathbf{y}^{(1)}; \mathcal{C}_1\big) \neq m_1 \\ &\qquad\quad \vee\, d\big(\mathbf{x}^{(1)}(m_1), \mathbf{y}^{(1)}\big) \geq n_1\,\Lambda_{1,\mathrm{c}} \\ &\qquad\quad \vee\, \exists m_1' \neq m_1 : d\big(\mathbf{x}^{(1)}(m_1'), \mathbf{y}^{(1)}\big) < n_1\,\Lambda_{1,\mathrm{w}} \Big\}. \end{aligned} \tag{24}$$

A triple $\big(R_1, \Lambda_{1,\mathrm{c}}, \Lambda_{1,\mathrm{w}}\big)$ is said to be achievable for Phase 1 if for every $\epsilon > 0$ there exists some positive integer $n_0(\epsilon)$ such that for every $n_1 \geq n_0(\epsilon)$ there exists a rate-R_1 blocklength-n_1 codebook statisfying that for every message m_1

$$\Pr\Big[\mathbf{Y}^{(1)} \in \mathcal{E}_{m_1}\big(\Lambda_{1,\mathrm{c}}, \Lambda_{1,\mathrm{w}}\big) \,\Big|\, M_1 = m_1\Big] < \epsilon. \tag{25}$$

Using a random coding argument, one can show that $\big(R_1, \Lambda_{1,\mathrm{c}}, \Lambda_{1,\mathrm{w}}\big)$ is achievable for Phase 1 whenever there exists some input distribution Q_1 such that

$$R_1 < I(Q_1, W), \tag{26a}$$

$$\Lambda_{1,\mathrm{c}} > \sum_{x \in \mathcal{X}} Q_1(x)\, W(y|x)\, d(x, y), \tag{26b}$$

and

$$\Lambda_{1,\mathrm{w}} < \min_{\substack{P_{XY}: P_X = Q_1 \\ P_Y = (Q_1 W) \\ I(X;Y) \leq R_1}} \mathsf{E}\big[d(X, Y)\big]. \tag{26c}$$

To specify the requirements from the second codebook $\mathcal{C}_2$, let $\mathcal{F}_{m_2}(\Lambda_{2,\mathrm{c}}, \Lambda_{\mathrm{F}}) \subseteq \mathcal{Y}^{n_2}$ comprise the Phase-2 output sequences $\mathbf{y}^{(2)}$ that result in the mismatched decoder producing a message other than m_2; or in Message m_2 accumulating a metric exceeding $n_2\,\Lambda_{2,\mathrm{c}}$; or in the metric accumulated by the all-x_{F} n_2-tuple being smaller than $n_2\,\Lambda_{\mathrm{F}}$

$$\begin{aligned} \mathcal{F}_{m_2}(\Lambda_{2,\mathrm{c}}, \Lambda_{\mathrm{F}}) = \Big\{ &\mathbf{y}^{(2)} \in \mathcal{Y}^{n_2} : \phi_d\big(\mathbf{y}^{(2)}; \mathcal{C}_2\big) \neq m_2 \\ &\vee\, d\big(\mathbf{x}^{(2)}(m_2), \mathbf{y}^{(2)}\big) \geq n_2\,\Lambda_{2,\mathrm{c}} \\ &\vee\, d\big(\mathbf{x}_{\mathrm{F}}, \mathbf{y}^{(2)}\big) \leq n_2\,\Lambda_{\mathrm{F}} \Big\}. \end{aligned} \tag{27}$$

A tuple $\big(R_2, \Lambda_{2,\mathrm{c}}, \Lambda_{\mathrm{F}}\big)$ is said to be achievable for Phase 2 if for every $\epsilon > 0$ there exists some positive integer $n_0(\epsilon)$ such that for every $n_2 \geq n_0(\epsilon)$ there exists a rate-R_2 blocklength-n_2 codebook statisfying that for every message m_2

$$\Pr\Big[\mathbf{Y}^{(2)} \in \mathcal{F}_{m_2}\big(\Lambda_{2,\mathrm{c}}, \Lambda_{\mathrm{F}}\big) \,\Big|\, M_2 = m_2\Big] < \epsilon. \tag{28}$$

We shall next provide a sufficient condition for $\big(R_2, \Lambda_{2,\mathrm{c}}, \Lambda_{\mathrm{F}}\big)$ to be achievable in terms of the *costrained mismatched capacity*. To this end, let us define the "reward" $g(x)$ of $x \in \mathcal{X}$ as the expected decoding metric associated with x_{F} when the transmitted symbol is x

$$g(x) = \sum_{y \in \mathcal{Y}} W(y|x)\, d(x_{\mathrm{F}}, y). \tag{29}$$

Let us extend this to length-n_2 sequences $\mathbf{x} \in \mathcal{X}^{n_2}$ additively

$$g(\mathbf{x}) = \sum_{k=1}^{n_2} g(x_k), \quad \mathbf{x} \in \mathcal{X}^{n_2}. \tag{30}$$

Let us also define the "cost" $h(x)$ of $x \in \mathcal{X}$ as the expected decoding metric associated with x when x is transmitted

$$h(x) = \sum_{y \in \mathcal{Y}} W(y|x)\, d(x, y), \tag{31}$$

and let us extend this to length-n_2 sequences additively

$$h(\mathbf{x}) = \sum_{k=1}^{n_2} h(x_k), \quad \mathbf{x} \in \mathcal{X}^{n_2}. \tag{32}$$

Denote by $C(W, d, \Lambda_{2,\text{c}}, \Lambda_{\text{F}})$ the supremum of achievable rates when we require a small maximal probability of error when decoding using ϕ_d and we impose the constraint that each codeword have a cost of at most $n_2\,\Lambda_{2,\text{c}}$ and a reward of at least $n_2\,\Lambda_{\text{F}}$. Notice that $\bigl(R_2, \Lambda_{2,\text{c}}, \Lambda_{\text{F}}\bigr)$ is achievable whenever $R_2 < C(W, d, \Lambda_{2,\text{c}} - \delta, \Lambda_{\text{F}} + \delta)$ for some $\delta > 0$. Indeed, by the law of large numbers (which can be invoked because the metric d is finite), if the transmitted n_2 tuples $\mathbf{x}^{(2)}$ satisfies the reward constraint $g\bigl(\mathbf{x}^{(2)}\bigr) \geq n_2\,(\Lambda_{\text{F}} + \delta)$, then

$$\lim_{n_2 \to \infty} \Pr\Bigl[d\bigl(\mathbf{x}_{\text{F}}, \mathbf{Y}^{(2)}\bigr) \leq n_2\,\Lambda_{\text{F}} \Bigm| \mathbf{X}^{(2)} = \mathbf{x}^{(2)}\Bigr] = 0. \tag{33}$$

Likewise, if the transmitted n_2 tuples $\mathbf{x}^{(2)}$ satisfies the cost constraint $h\bigl(\mathbf{x}^{(2)}\bigr) \leq n_2\,(\Lambda_{2,\text{c}} - \delta)$, then

$$\lim_{n_2 \to \infty} \Pr\Bigl[d\bigl(\mathbf{x}^{(2)}, \mathbf{Y}^{(2)}\bigr) \geq n_2\,\Lambda_{2,\text{c}} \Bigm| \mathbf{X}^{(2)} = \mathbf{x}^{(2)}\Bigr] = 0. \tag{34}$$

Calculating $C(W, d, \Lambda_{2,\text{c}}, \Lambda_{\text{F}})$ is difficult, because we do not even have a single-letter expression for the unconstrained mismatch capacity. But we can mimick the random-coding arguments of [1], [2] to lower-bound this expression:

$$C(W, d, g, \Lambda_{2,\text{c}}, \Lambda_{\text{F}}) \geq C_{\text{LM}}(W, d, \Lambda_{2,\text{c}}, \Lambda_{\text{F}}), \tag{35}$$

where

$$C_{\text{LM}}(W, d, \Lambda_{2,\text{c}}, \Lambda_{\text{F}}) = \max_{\substack{\sum_{x \in \mathcal{X}} Q_2(x)\,g(x) \geq \Lambda_{\text{F}} \\ \sum_{x \in \mathcal{X}} Q_2(x)\,h(x) \leq \Lambda_{2,\text{c}}}} I_{\text{LM}}(Q_2), \tag{36}$$

and

$$I_{\text{LM}}(Q_2) = \min_{\substack{P_X = Q_2 \\ P_Y = (Q_2 W) \\ \sum_{x,y} P_{X,Y}(x,y)\,d(x,y) \\ \leq \sum_{x,y} Q_2(x)\,W(y|x)\,d(x,y)}} I(X;Y). \tag{37}$$

We now look at the two phases together and argue that if

$$n_1\,\Lambda_{1,\text{c}} + n_2\,\Lambda_{2,\text{c}} \leq n_1\,\Lambda_{1,\text{w}} + n_2\,\Lambda_{\text{F}}, \tag{38}$$

then

$$\phi_d(\mathbf{y}) = (m_1, m_2) \tag{39}$$

whenever the output sequence $\mathbf{y} = \bigl(\mathbf{y}^{(1)}, \mathbf{y}^{(2)}\bigr)$ is such that

$$\mathbf{y}^{(1)} \notin \mathcal{E}_{m_1}\bigl(\Lambda_{1,\text{c}}, \Lambda_{1,\text{w}}\bigr) \tag{40a}$$

and

$$\mathbf{y}^{(2)} \notin \mathcal{F}_{m_2}(\Lambda_{2,\text{c}}, \Lambda_{\text{F}}). \tag{40b}$$

To see why, consider first the metric accumulated by (m_1, m_2). This can be expressed, by (20) and (23), as

$$\begin{aligned} &d\bigl(\mathbf{x}(m_1, m_2, \mathbf{y}^{(1)}), \mathbf{y}\bigr) \\ &\quad = d\bigl(\mathbf{x}^{(1)}(m_1), \mathbf{y}^{(1)}\bigr) + d\bigl(\mathbf{x}^{(2)}(m_2), \mathbf{y}^{(2)}\bigr) && (41) \\ &\quad < n_1\,\Lambda_{1,\text{c}} + n_2\,\Lambda_{2,\text{c}}, && (42) \end{aligned}$$

where the inequality follow from (40).

Next consider the metric accumulated by (m_1', m_2') where $m_1' \neq m_1$ (and m_2' is either equal to m_2 or not). In this case (40) implies that

$$\begin{aligned} &d\bigl(\mathbf{x}(m_1', m_2', \mathbf{y}^{(1)}), \mathbf{y}\bigr) \\ &\quad = d\bigl(\mathbf{x}^{(1)}(m_1'), \mathbf{y}^{(1)}\bigr) + d\bigl(\mathbf{x}_{\text{F}}, \mathbf{y}^{(2)}\bigr) && (43) \\ &\quad > n_1\,\Lambda_{1,\text{w}} + n_2\,\Lambda_{\text{F}}, && (44) \\ &\quad \geq d\bigl(\mathbf{x}(m_1, m_2, \mathbf{y}^{(1)}), \mathbf{y}\bigr) && (45) \end{aligned}$$

where the first equality follows from the assumption that $m_1' \neq m_1$, from (40a), and from (23); the first inequality follows from (40); and the final inequality follows from (42) and (38).

Finally, consider the the metric accumulated by (m_1, m_2') when m_2' differs from m_2. In this case (40) implies that

$$\begin{aligned} &d\bigl(\mathbf{x}(m_1, m_2', \mathbf{y}^{(1)}), \mathbf{y}\bigr) \\ &\quad = d\bigl(\mathbf{x}^{(1)}(m_1), \mathbf{y}^{(1)}\bigr) + d\bigl(\mathbf{x}^{(2)}(m_2'), \mathbf{y}^{(2)}\bigr) && (46) \\ &\quad > d\bigl(\mathbf{x}^{(1)}(m_1), \mathbf{y}^{(1)}\bigr) + d\bigl(\mathbf{x}^{(2)}(m_2), \mathbf{y}^{(2)}\bigr) && (47) \\ &\quad = d\bigl(\mathbf{x}(m_1, m_2, \mathbf{y}^{(1)}), \mathbf{y}\bigr) && (48) \end{aligned}$$

where (47) holds because (40b) implies $\phi_d\bigl(\mathbf{y}^{(2)}; \mathcal{C}_2\bigr) = m_2$.

The above derivation allows us to conclude the following.

Theorem V.1. *Let $W(y|x)$ be a DMC and $d(x,y)$ a finite decoding metric. Let Q_1 and Q_2 be input distributions, $x_{\text{F}} \in \mathcal{X}$ an input symbol, and suppose $R_1 < I(Q_1, W)$ and $R_2 < I_{\text{LM}}(Q_2)$. Let the thresholds $\Lambda_{1,\text{c}}$, $\Lambda_{1,\text{w}}$ satisfy* (26b)–(26c)*, and let $\Lambda_{2,\text{c}}$, Λ_{F} satisfy the constraints in* (36)*. Then the rate $\alpha I(Q_1; W) + (1-\alpha) I_{\text{LM}}(Q_2)$ is achievable whenver the constant $0 < \alpha < 1$ is such that*

$$\alpha\,\Lambda_{1,\text{c}} + (1-\alpha)\,\Lambda_{2,\text{c}} \leq \alpha\,\Lambda_{1,\text{w}} + (1-\alpha)\,\Lambda_{\text{F}}. \tag{49}$$

Acknowledgment

The author thanks Christoph Bunte for his comments.

References

[1] I. Csiszár and J. Körner, "Graph decomposition: a new key to coding theorems," *IEEE Trans. on Inform. Theory*, vol. 27, no. 1, pp. 5–12, Jan. 1981.

[2] J. Y. N. Hui, "Fundamental issues of multiple accessing," Ph.D. dissertation, Massachusetts Institute of Technology, 1983.

[3] A. Lapidoth, "Mismatched decoding and the multiple-access channel," *IEEE Trans. on Inform. Theory*, vol. 42, no. 5, pp. 1439–1452, Sept. 1996.

[4] I. Csiszár and P. Narayan, "Channel capacity for a given decoding metric," *IEEE Trans. on Inform. Theory*, vol. 41, no. 1, pp. 35–43, Jan. 1995.

[5] J. Scarlett, A. Martinez, and A. Guillén i Fàbregas, "Mismatched decoding: Error exponents, second-order rates and saddlepoint approximations," *IEEE Trans. on Inform. Theory*, vol. 60, no. 5, pp. 2647–2666, May 2014.

[6] A. Somekh-Baruch, "On achievable rates and error exponents for channels with mismatched decoding," *IEEE Trans. Inform. Theory*, vol. 61, no. 2, pp. 727–740, 2015.

[7] J. Scarlett, A. Martinez, and A. Guillén i Fàbregas, "Multiuser random coding techniques for mismatched decoding," *IEEE Trans. on Inform. Theory*, vol. 62, no. 7, pp. 3950–3970, July 2016.

[8] B. Nakiboğlu and L. Zheng, "Errors-and-erasures decoding for block codes with feedback," *IEEE Trans. on Inform. Theory*, vol. 58, no. 1, pp. 24–49, Jan. 2012.

[9] C. Bunte and A. Lapidoth, "The zero-undetected-error capacity of discrete memoryless channels with feedback," in *Proceedings 50th Annual Allerton Conference on Communication, Control, and Computing, Monticello, IL, Oct. 1–5, 2012.*