A Necessary Condition for the Transmissibility of Correlated Sources over a MAC

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***Abstract*—A necessary condition for the transmissibility of correlated sources over a multi-access channel (MAC) is presented. The condition is related to Wyner's common information and to the Slepian-Wolf capacity region of the MAC with private and common messages. An analogous condition for the transmissibility of remote sources over a MAC is also derived. Here the transmitters only observe noisy versions of the sources.**

I. Introduction and Setup

We consider the setup in Figure 1 of a two-to-one discrete memoryless multiple-access channel (MAC) with finite input alphabets $\mathcal{X}_1$ and $\mathcal{X}_2$, finite output alphabet $\mathcal{Y}_2$, and transition law $P_{Y|X_1X_2}$. The channel is used in order to enable the

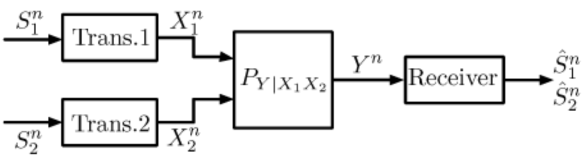

Fig. 1. Transmission of a remote source over a two-user MAC.

receiver to reconstruct, with some required fidelity, the two source sequences

$$S_1^n := (S_{1,1}, \ldots, S_{1,n}) \qquad \text{and} \qquad S_2^n := (S_{2,1}, \ldots, S_{2,n}),$$

where the pairs $\{(S_{1,t}, S_{2,t})\}_{t=1}^n$ are drawn IID from the finite set $\mathcal{S}_1 \times \mathcal{S}_2$ according to the joint source distribution $P_{S_1S_2}$. Transmitter i observes the sequence S_i^n and generates its channel inputs $X_i^n := (X_{i,1}, \ldots, X_{i,n})$ as

$$X_i^n = f_i^{(n)}(S_i^n), \qquad i \in \{1, 2\}, \tag{1}$$

for some encoding function $f_i^{(n)} \colon \mathcal{S}_i^n \to \mathcal{X}_i^n, \quad i \in \{1, 2\}$. The receiver produces the estimates $\hat{S}_1^n := (\hat{S}_{1,1}, \ldots, \hat{S}_{1,n})$ and $\hat{S}_2^n := (\hat{S}_{2,1}, \ldots, \hat{S}_{2,n})$ based on the channel outputs $Y^n := (Y_1, \ldots, Y_n)$. Thus,

$$\begin{pmatrix} \hat{S}_1^n \\ \hat{S}_2^n \end{pmatrix} = g^{(n)}(Y^n), \tag{2}$$

where the decoding function is of the form $g^{(n)} \colon \mathcal{Y}^n \to \hat{\mathcal{S}}_1^n \times \hat{\mathcal{S}}_2^n$, and where $\hat{\mathcal{S}}_1$ and $\hat{\mathcal{S}}_2$ denote the finite reconstruction alphabets.

Given two nonnegative distortion functions

$$d_i \colon \mathcal{S}_i \times \hat{\mathcal{S}}_i \to \mathbb{R}_+, \qquad i \in \{1, 2\},$$

(where $\mathbb{R}_+$ denotes the nonnegative reals) and two maximum-allowed distortions $D_1, D_2 \geq 0$, we require that

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\big[d_1(S_{1,t}, \hat{S}_{1,t})\big] \leq D_1, \tag{3a}$$

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\big[d_2(S_{2,t}, \hat{S}_{2,t})\big] \leq D_2. \tag{3b}$$

Given distortion functions d_1 and d_2, we say that the *source-channel pair* $(P_{S_1S_2},\ P_{Y|X_1X_2})$ *is* (D_1, D_2)*-feasible* if for each blocklength n it is possible to find encoding functions $f_1^{(n)}$ and $f_2^{(n)}$, and a reconstruction function $g^{(n)}$ such that (3) holds. Our interest is in characterizing the pairs (D_1, D_2) that are feasible.

A special case of this problem was studied by Lapidoth and Tinguely [1] who considered a bivariate Gaussian source; a power-limited Gaussian MAC; and the squared-error distortion functions.

Another special case is the *lossless case*[1] where the distortion functions are Hamming distortions and the maximum allowed distortions are zero:

$$d_i \colon (s_i, \hat{s}_i) \mapsto \begin{cases} 1, & \hat{s}_i \neq s_i \\ 0, & \hat{s}_i = s_i \end{cases}, \qquad i \in \{1, 2\}, \tag{4a}$$

and

$$D_1 = D_2 = 0. \tag{4b}$$

We say that a source-channel pair is *feasible in the lossless case* if it is $(0, 0)$-feasible in this setting.

Cover, El Gamal, and Salehi [2] (for the lossless case), Salehi [3] and Minero, Lim, and Kim [4] (both for the lossy case) presented sufficient conditions for a source-channel pair $(P_{S_1S_2},\ P_{Y|X_1X_2})$ to be (D_1, D_2)-feasible. Here we present necessary conditions. These do not, in general, coincide with the sufficient conditions.

Necessary conditions for the lossless case were previously derived by Kang and Ulukus [5] by generalizing the necessary condition of Lapidoth and Tinguely [1], which is based on the observation that when the source is a bivariate Gaussian, the correlation coefficient between the MAC inputs cannot exceed the correlation coefficient between the source components.

[1]The term *lossless source coding* is traditionally used for a slightly different scenario where the *probability of blockerror* $\hat{S}_i^n \neq S_i^n$ is required to tend to 0; specializing Condition (3) to (4) implies that the *average probability of symbol error* tends to 0. Our condition is thus stronger, and as a consequence, any necessary condition for feasibility that we present for our lossless setup is also necessary condition for feasibility in the traditional lossless setup.

Our necessary condition for the lossless case (Corollary 1.2) is difficult to compare to Kang and Ulukus's condition [5], but it does seem to be easier to verify, especially when the source has a known rate-distortion function and the MAC has a known Slepian-Wolf capacity region for private and common messages [6]; see Remark 1.

In Section III we consider a more general setup and propose a necessary condition for the transmissibility of *remote* sources over a MAC. This setup differs from our original setup in that each transmitter only observes a *noisy version* of its source component. Special cases of this setup were previously studied and solved by Gastpar [7], by Lapidoth and Wang [8], and by Tian, Chen, Diggavi, and Shamai [9].

II. Main Results

A. The General Lossy Case

Theorem 1: Fix distortion functions d_1 and d_2. If the source-channel pair $(P_{S_1S_2},\ P_{Y|X_1,X_2})$ is (D_1, D_2)-feasible, then *for every* auxiliary random variable (RV) W forming a Markov chain with the source components,

$$S_1 \to W \to S_2, \tag{5}$$

there exists an auxiliary RV U forming a Markov chain with the inputs

$$X_1 \to U \to X_2 \tag{6}$$

and two reconstruction symbols $\hat{S}_1$ and $\hat{S}_2$ such that the following five constraints (7) are satisfied:

$$I(S_1; \hat{S}_1) \le I(X_1; Y|X_2, U) + I(S_1; W) \tag{7a}$$
$$I(S_2; \hat{S}_2) \le I(X_2; Y|X_1, U) + I(S_2; W) \tag{7b}$$
$$I(S_1, S_2; \hat{S}_1, \hat{S}_2) \le I(X_1, X_2; Y|U) + I(S_1, S_2; W) \tag{7c}$$
$$I(S_1, S_2; \hat{S}_1, \hat{S}_2) \le I(X_1, X_2; Y), \tag{7d}$$

and

$$\mathbb{E}\big[d_i(S_i, \hat{S}_i)\big] \le D, \qquad i \in \{1, 2\}. \tag{7e}$$

Proof: Follows by specializing Theorem 2 in Section IV to $T_1 = S_1$ and $T_2 = S_2$. ∎

Remark 1:

1) Every choice of the auxiliary RV W that satisfies (5) yields a necessary condition. An interesting choice for symmetric settings is Wyner's common part [10]. (See Corollary 1.1 for more details.) With this choice, $I(S_1, S_2; W)$ equals Wyner's common information $C_{\text{Wyner}}(S_1, S_2)$ in (8).
2) The choice of the conditional law $P_{\hat{S}_1,\hat{S}_2|S_1,S_2}$ affects only the left-hand sides (LHS) of (7a)–(7d) and the distortion constraints (7e). For various sources and distortion functions, it is possible to identify subsets of conditional distributions satisfying (7e) to which one can restrict attention when evaluating the infeasibility condition in Theorem 1. For example, for a bivariate Gaussian source and squared-error distortion functions it suffices to consider conditional laws $P_{\hat{S}_1\hat{S}_2|S_1S_2}$ that result in $(\hat{S}_1, \hat{S}_2)$ being jointly Gaussian with the source (S_1, S_2).
3) The joint law $P_{UX_1X_2}$ should be chosen to maximize the right-hand sides (RHS) of (7a)–(7d) subject to (6). These coincide with the RHSs of the rate-constraints in Slepian and Wolf's capacity region of the MAC with private and common messages [6]. Our necessary condition is thus particularly simple to evaluate for channels, such as the Gaussian MAC [11], whose Slepian-Wolf capacity region is known.

We obtain a simpler—albeit generally weaker—necessary condition, if in Theorem 1 we relax the "single-rate" constraints (7a) and (7b). To state the resulting corollary in a compact form, we make the following two definitions. Let C_{Wyner} denote *Wyner's common information* [10]:

$$C_{\text{Wyner}}(S_1, S_2) := \min_{S_1 \to W \to S_2} I(S_1, S_2; W). \tag{8}$$

Let $R_{S_1S_2}(D_1, D_2)$ denote the standard rate-distortion function when compressing the bivariate source sequence (S_1^n, S_2^n) so as to satisfy the two distortion constraints (3):

$$R_{S_1S_2}(D_1, D_2) := \min\ I(S_1, S_2; \hat{S}_1, \hat{S}_2), \tag{9}$$

where the minimum is over all reconstruction random variables $\hat{S}_1$ and $\hat{S}_2$ that satisfy (7e).

Corollary 1.1: If the pair $(P_{S_1S_2},\ P_{Y|X_1,X_2})$ is (D_1, D_2)-feasible then

$$\begin{aligned} &R_{S_1S_2}(D_1, D_2) \\ &\le \max_{X_1 \to U \to X_2} \min\big\{I(X_1, X_2; Y|U) + C_{\text{Wyner}}(S_1, S_2), \\ &\qquad\qquad\qquad\qquad I(X_1, X_2; Y)\big\}. \end{aligned} \tag{10}$$

Example 1: Consider a bivariate Gaussian source

$$\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, Q\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right), \quad Q > 0, \tag{11a}$$

and a memoryless additive Gaussian noise MAC

$$Y = X_1 + X_2 + Z, \tag{11b}$$

whose inputs X_1 and X_2 are block-power constrained to the same power P, and where Z is a standard Gaussian. Let $d_1, d_2\colon (s, \hat{s}) \mapsto (s - \hat{s})^2$ be squared-error distortion functions and $D_1 = D_2 = D$.

Let us evaluate the necessary condition of Corollary 1.1 for this example.[2] For this source, $R_{S_1S_2}(D_1, D_2)$ and $C_{\text{Wyner}}(S_1, S_2)$ are well known [10] and

$$C_{\text{Wyner}}(S_1, S_2) = \frac{1}{2}\log_2 \frac{1+\rho}{1-\rho}. \tag{12}$$

Moreover, according to the reasoning in [11, 12], we can restrict to jointly Gaussian triples (U, X_1, X_2) where X_1 and X_2 are of full power P. We thus obtain the following necessary condition: If the source-channel pair in (11) is (D, D)-feasible, then the source parameters ρ and Q, the channel input-power P, and the maximum allowed distortion D have to satisfy Condition (13) on the next page.

[2] We derived our results for finite sources and discrete channels without input-cost constraints. They extend however in a straight-forward manner to the setup in this example.

$$R_{S_1S_2}(D,D) \le \frac{1}{2}\log_2\left(1+2P\left(1+\frac{-(1-\rho)+\sqrt{(1-\rho)^2+4(1+\rho)\left(2\rho+\frac{\rho}{P}\right)}}{2(1+\rho)}\right)\right) \tag{13}$$

The necessary condition of [1] is stronger than ours and is tight in the high-SNR regime. It is obtained if in (13) we replace the term

$$\frac{-(1-\rho)+\sqrt{(1-\rho)^2+4(1+\rho)\left(2\rho+\frac{\rho}{P}\right)}}{2(1+\rho)} \tag{14}$$

by the smaller term ρ. See Figure 2 for a comparison of the two terms when $P = 10$. However, the Lapidoth-Tinguely condition is tailored to the Gaussian source-channel pair, whereas our condition in Theorem 1 holds for general sources and channels.

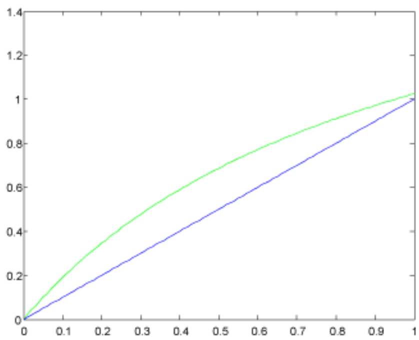

Fig. 2. The lower blue line shows the mapping $\rho \mapsto \rho$ and the upper green line shows the mapping from ρ to the expression in (14). Power $P = 10$.

B. The Lossless Case

In the lossless case, Theorem 1 specializes to the following:

Corollary 1.2: If the source-channel pair $(P_{S_1S_2},\ P_{Y|X_1X_2})$ is feasible in the lossless case, then for every auxiliary RV W forming the Markov chain

$$S_1 \to W \to S_2 \tag{15}$$

there exists an auxiliary RV U satisfying

$$X_1 \to U \to X_2 \tag{16}$$

and the following four conditions:

$$H(S_1|S_2) \le I(X_1;Y|X_2,U)+I(S_1;W|S_2) \tag{17a}$$

$$H(S_2|S_1) \le I(X_2;Y|X_1,U)+I(S_2;W|S_1) \tag{17b}$$

$$\begin{aligned} H(S_1|S_2)+H(S_2|S_1) &\le I(X_1,X_2;Y|U) \\ &\quad +I(S_1;W|S_2)+I(S_2;W|S_1) \\ H(S_1,S_2) &\le I(X_1,X_2;Y). \end{aligned} \tag{17c}$$

Example 2 (DSBS(q) source and Gaussian MAC): Let (S_1,S_2) be a *doubly-symmetric binary source of parameter q (DSBS(q))*, i.e., S_1 and S_2 are Bernoulli-$1/2$ random variables and $\Pr[S_1 \neq S_2] = q$. Let the MAC be as in Example 1.

For simplicy we again relax constraints (17a) and (17b). As in Corollary 1.1, the strongest condition is obtained when W is Wyner's common part and $I(S_1,S_2;W)$ is hence Wyner's common information. For the DSBS(q) in this example [10]

$$C_{\text{Wyner}}(S_1,S_2) = 1 + H_b(q) - 2H_b(\gamma),$$

where $\gamma = \frac{1}{2}\left(1-\sqrt{1-2q}\right)$ and $H_b(\cdot)$ denotes the binary entropy function.

By the arguments in [11, 12], we can restrict ourselves to jointly Gaussian triples (U,X_1,X_2), where X_1 and X_2 are of full power P. Optimizing over this joint Gaussian distribution, we obtain the following necessary condition.

If the described source-channel pair $(P_{S_1S_2},\ P_{Y|X_1X_2})$ is feasible for the lossless case, then the source-parameter q and the channel input-power P must satisfy

$$\begin{aligned} &1 + H_b(q) \\ &\le \frac{1}{2}\log_2\left(1+P\left(2+\frac{\sqrt{1+4\beta(\beta-1)(1+\frac{1}{2P})}-1}{\beta}\right)\right), \end{aligned} \tag{18}$$

where $\beta := 2^{2(1+H_b(q)-2H_b(\gamma))}$.

Notice that the LHS of Condition (18) is strictly increasing in $q \in [0,\frac{1}{2}]$ and its RHS is strictly decreasing. Moreover, for $P < \frac{3}{4}$ Condition (18) is violated even for $q = 0$. Thus, irrespective of the source parameter $q \in [0,\frac{1}{2}]$, the DSBS(q) cannot be sent over the Gaussian MAC with input powers $P < \frac{3}{4}$. For $P \ge \frac{3}{4}$ Condition (18) is satisfied for $q = 0$, which allows us to define $q_{\sup}$ as the supremum over all $q \in [0,\frac{1}{2}]$ such that Condition (18) holds. Our necessary condition states that for all $q \in (q_{\sup},\frac{1}{2}]$, the DSBS($q$) cannot be sent over the Gaussian MAC with input powers P. Numerically we find:

P	1	2	5
$q_{\sup}$	0.002	0.115	0.373

III. Transmission of Remote Sources over a MAC

A. Setup

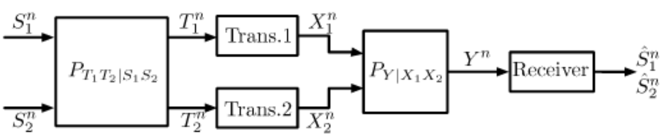

Fig. 3. Transmission of remote sources over a two-user MAC.

We now consider a setup (Figure 3) where the transmitters cannot directly observe the source sequences S_1^n and S_2^n, but only the noisy versions $T_1^n := (T_{1,1},\ldots,T_{1,n})$ and $T_2^n := (T_{2,1},\ldots,T_{2,n})$, respectively. For each $t \in \{1,\ldots,n\}$, the pair $(T_{1,t},\ T_{2,t})$ takes values in the finite set $\mathcal{T}_1 \times \mathcal{T}_2$ and is generated by the memoryless channel $P_{T_1T_2|S_1S_2}$ from the source pair $(S_{1,t}, S_{2,t})$. The joint PMF of (T_1, T_2, S_1, S_2) is thus $P_{S_1S_2}\times, P_{T_1T_2|S_1S_2}$. Each transmitter generates its channel inputs X_i^n as a function of its observed symbols T_i^n:

$$X_i^n = f_i^{(n)}(T_i^n), \qquad i \in \{1,2\}, \tag{19}$$

for some encoding function $f_i^{(n)} \colon \mathcal{T}_i^n \to \mathcal{X}_i^n, \qquad i \in \{1,2\}$. The receiver acts in the same manner as before. We say that the *source-channels triple* $(P_{S_1S_2},\ P_{T_1T_2|S_1S_2},\ P_{Y|X_1X_2})$ *is* (D_1, D_2)*-feasible* if it is possible to find encoding functions $\{f_i^{(n)}\}_{n=1}^{\infty}$, for $i \in \{1,2\}$, and a reconstruction function $\{g^{(n)}\}_{n=1}^{\infty}$ such that (3) holds.

The special case with a bivariate Gaussian source that is observed in Gaussian noise, with a power-limited Gaussian MAC, and with squared-error distortion functions was studied by Lapidoth and Wang [8], see also [9]. Gastpar derived a condition that is sufficient and necessary for another special case with a single Gaussian source [7] and where the two transmitters have equally-noisy observations and equal power. The corresponding asymmetric setup was partially solved by Tian, Chen, Diggavi, and Shamai [9].

B. Results and Example

Theorem 2: Fix distortion functions d_1 and d_2. If the source-channels triple $(P_{S_1S_2},\ P_{T_1T_2|S_1S_2},\ P_{Y|X_1X_2})$ is (D_1, D_2)-feasible, then *for every* auxiliary RV W forming the Markov chain

$$T_1 \to W \to T_2, \tag{20}$$

there exists an auxiliary RV U forming a Markov chain with the channel inputs,

$$X_1 \to U \to X_2, \tag{21}$$

and a pair $(\hat{S}_1, \hat{S}_2)$ so that

$$I(S_1; \hat{S}_1) \le I(X_1; Y|X_2, U) + I(S_1; T_2, W) \tag{22a}$$

$$I(S_2; \hat{S}_2) \le I(X_2; Y|X_1, U) + I(S_2; T_1, W) \tag{22b}$$

$$I(S_1, S_2; \hat{S}_1, \hat{S}_2) \le I(X_1, X_2; Y|U) + I(S_1, S_2; W) \tag{22c}$$

$$I(S_1, S_2; \hat{S}_1, \hat{S}_2) \le I(X_1, X_2; Y), \tag{22d}$$

and

$$\mathbb{E}\big[d_i(S_i, \hat{S}_i)\big] \le D_i, \qquad i \in \{1,2\}. \tag{22e}$$

Proof: See Section IV. ■

A special case of interest is a single source

$$S_1 = S_2 = S \tag{23a}$$

where the receiver produces a single reconstruction, so

$$d_1 = d_2 = d \qquad \text{and} \qquad D_1 = D_2 = D. \tag{23b}$$

Gastpar's [7] joint source-channel version of the Gaussian CEO problem is a special case of this scenario.

To apply Theorem 2 to this setting, let us denote by $R_S(D)$ the rate-distortion function

$$R_S(D) := \min I(S; \hat{S}), \tag{24}$$

where the minimum is over all reconstructions $\hat{S}$ such that $\mathbb{E}[d(S, \hat{S})] \le D$.

Corollary 2.1: Consider the special case in (23) and let a distortion function d be given. If the source-channels triple $(P_{SS},\ P_{T_1T_2|S},\ P_{Y|X_1X_2})$ is (D, D)-feasible, then for every auxiliary RV W forming the Markov chain (20) there exists an auxiliary RV U forming the Markov chain (21) and a reconstruction $\hat{S}$ so that:

$$R_S(D) \le I(X_1; Y|X_2, U) + I(S; T_2, W) \tag{25a}$$

$$R_S(D) \le I(X_2; Y|X_1, U) + I(S; T_1, W) \tag{25b}$$

$$R_S(D) \le I(X_1, X_2; Y|U) + I(S; W) \tag{25c}$$

$$R_S(D) \le I(X_1, X_2; Y). \tag{25d}$$

Example 3: Consider a zero-mean Gaussian source S of variance $Q > 0$. The transmitters observe

$$T_1 = (\tilde{T}_1, E) \quad \text{and} \quad T_2 = (\tilde{T}_2, E), \tag{26}$$

where E is a Bernoulli-$1/2$ RV independent of the source S and where

$$\tilde{T}_1 := \begin{cases} S + V + S_0, & \text{if } E = 0 \\ S_0, & \text{if } E = 1 \end{cases} \tag{27}$$

and

$$\tilde{T}_2 := \begin{cases} S_0, & \text{if } E = 0 \\ S + V + S_0, & \text{if } E = 1, \end{cases} \tag{28}$$

for S_0 and V zero-mean Gaussians of variances Q and $\sigma_V^2 > 0$ and independent of each other and of the pair (E, S). The distortion function d is the squared-error distortion function. As in the previous examples we consider a memoryless Gaussian MAC of unit noise-variance and equal input-powers P.

We evaluate Corollary 2.1 for the described setup. For our Gaussian source, $R_S(D) = \frac{1}{2}\log_2^+\left(\frac{Q}{D}\right)$, where $\log_2^+(x) := \max\{0, x\}$. We choose $W = (S_0, E)$, which satisfies Markov chain (20) because $I(T_1; T_2|W) = 0$.

Since $I(S; W) = 0$ and since $I(X_1, X_2; Y|U)$ cannot exceed the sum-rate capacity of the Gaussian MAC with private messages, namely $\frac{1}{2}\log_2(1 + 2P)$, Constraint (25c) is equivalent to

$$\frac{1}{2}\log_2^+\left(\frac{Q}{D}\right) \le \frac{1}{2}\log_2(1 + 2P). \tag{29a}$$

On the other hand, since $I(S; T_2, W) = \frac{1}{4}\log_2\left(1 + \frac{Q}{\sigma_v^2}\right)$ and since $I(X_1; Y|U, X_2)$ cannot exceed the capacity of the Gaussian point-to-point channel from Transmitter 1 to the receiver, i.e., $\frac{1}{2}\log_2(1 + P)$, Constraint (25a) is equivalent to

$$\frac{1}{2}\log_2^+\left(\frac{Q}{D}\right) \le \frac{1}{2}\log_2(1 + P) + \frac{1}{4}\log_2\left(1 + \frac{Q}{\sigma_v^2}\right). \tag{29b}$$

Constraints (25b) and (25d) are redundant.

We obtain the following necessary condition: If the described source-channels triple $(P_S,\ P_{T_1T_2|S},\ P_{Y|X_1X_2})$ is (D,D)-feasible, then the source variance Q, the channel input power P, and the distortion D must satisfy Conditions (29).

Bound (29b) is active when σ_v^2 is large and T_1 and T_2 are very noisy observations of the source S. Bound (29a) can be intuitively understood as saying that T_1 and T_2 have no common part related to the source S that could allow the MAC transmitters to cooperate in a useful manner.

IV. Proof of Theorem 2

Let distortion functions d_1 and d_2 and two maximum allowed distortions $D_1, D_2 \geq 0$ be given. Suppose a source-channels triple $(P_{S_1S_2},\ P_{T_1T_2|S_1S_2},\ P_{Y|X_1X_2})$ that is (D_1,D_2)-feasible. For each blocklength n, choose encoding and reconstruction functions $f_1^{(n)}$, $f_2^{(n)}$, and $g^{(n)}$ so that (3) holds. Choose probability laws $P_{W|S_1S_2T_1T_2}$ and P_W so that

$$P_{S_1S_2T_1T_2} \times P_{W|S_1S_2T_1T_2} = P_W P_{T_1|W} P_{T_2|W} P_{S_1S_2|T_1T_2W}.$$

Fix now a blocklength n, and let X_1^n, X_2^n, Y^n, $\hat{S}_1^n$, and $\hat{S}_2^n$ denote the channel-input, channel-output, and reconstruction sequences corresponding to the chosen $f_1^{(n)}$, $f_2^{(n)}$, and $g^{(n)}$.

For each $t \in \{1,\ldots,n\}$, generate the RV W_t as the output of a channel with transition law $P_{W|S_1S_2T_1T_2}$ and inputs $S_{1,t}, S_{2,t}, T_{1,t}, T_{2,t}$. Define $U_t := W^n$ and let Z be a uniform RV over $\{1,\ldots,n\}$ that is independent of all other random variables such as source symbols, inputs, outputs, etc. Define $W := W_Z$, $U := (W^n, Z)$, $Y := Y_Z$, and for $i \in \{1,2\}$, $S_i := S_{i,Z}$, $T_i := T_{i,Z}$, $X_i := X_{i,Z}$, $\hat{S}_i = \hat{S}_{i,Z}$.

We now proceed to prove constraints (22a)–(22d). Constraint (22a) is obtained as follows:

$$\begin{aligned}
I(S_1;\hat{S}_1) &\overset{(a)}{\leq} I(S_1;\hat{S}_1|Z) \overset{(b)}{=} \frac{1}{n}\sum_{t=1}^{n} I(S_{1,t};\hat{S}_{1,t}) \\
&\overset{(c)}{\leq} \frac{1}{n}\sum_{t=1}^{n} I(S_{1,t};\hat{S}_{1,t}|S_1^{t-1}) \\
&\leq \frac{1}{n}\sum_{t=1}^{n} I(S_{1,t};\hat{S}_1^{n}|S_1^{t-1}) = \frac{1}{n} I(S_1^n;\hat{S}_1^n) \\
&\overset{(d)}{\leq} \frac{1}{n} I(S_1^n;Y^n) \leq \frac{1}{n} I(S_1^n;Y^n,T_2^n,W^n) \\
&= \frac{1}{n} I(S_1^n;Y^n|T_2^n,W^n) + \frac{1}{n} I(S_1^n;T_2^n,W^n) \\
&\overset{(e)}{=} \frac{1}{n}\sum_{t=1}^{n} I(S_1^n;Y_t|T_2^n,W^n,Y^{t-1}) + I(S_1;T_2,W) \\
&\overset{(f)}{\leq} \frac{1}{n}\sum_{t=1}^{n} I(X_{1,t};Y_t|X_{2,t},U_t) + I(S_1;T_2,W) \\
&= I(X_{1,Z};Y|X_{2,Z},U_Z,Z) + I(S_1;T_2,W) \\
&= I(X_1;Y|X_2,U) + I(S_1;T_2,W), \qquad (30)
\end{aligned}$$

where (a) holds because Z is independent of S_1; (b) holds by the definition of $S_1, \hat{S}_1, Z$; (c) because S_1^n is i.i.d.; (d) by the data-processing inequality; (e) because (S_1^n, T_1^n, W^n) are i.i.d. and by the chain rule of mutual information; (f) because conditioning can only reduce entropy, because $X_{2,t}$ can be computed as a function from T_2^n and because of the Markov chain $(T_2^n, W^n, Y^{t-1}) \to (X_{1,t}, X_{2,t}) \to Y_t$.

Constraints (22b) and (22c) are obtained in a similar way. Related steps also allow us to prove Constraint (22d):

$$\begin{aligned}
I(S_1,S_2;\hat{S}_1,\hat{S}_2) &\leq I(S_1^n,S_2^n;Y^n) \\
&= \sum_{t=1}^{n} I(S_1^n,S_2^n;Y_t|Y^{t-1}) \\
&\leq \sum_{t=1}^{n} I(X_{1,t},X_{2,t};Y_t) \\
&= I(X_{1,Z},X_{2,Z};Y|Z) \\
&\leq I(X_1,X_2;Y). \qquad (31)
\end{aligned}$$

Markov chain (21) holds because $T_1^n \to (W^n, Z) \to T_2^n$ and because $X_{1,t}$ and $X_{2,t}$ are functions of T_1^n and T_2^n. Moreover, for $i \in \{1,2\}$:

$$\begin{aligned}
\frac{1}{n}\sum_{t=1}^{n} \mathbb{E}\big[d_i(S_{i,t},\hat{S}_{i,t})\big] &= \mathbb{E}_Z\Big[\mathbb{E}\big[d_i(S_i,\hat{S}_i)|Z\big]\Big] \\
&= \mathbb{E}\big[d_i(S_i,\hat{S}_i)\big].
\end{aligned}$$

Thus, given (3), for arbitrary $\epsilon > 0$ and if n is sufficiently large, $\mathbb{E}\big[d_i(S_i,\hat{S}_i)\big] \leq D_i + \epsilon$, for $\in \{1,2\}$.

Letting $\epsilon \to 0$ and continuity arguments conclude the proof.

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